Physics 501-22
Assignment 2

1)Given two spin 1/2 (ie, two level) systems, one of the systems is supposed to represent the states of an interferometer, the other a measuring apparatus. The interferometer has two input arms in which a particle enters onto a half silvered mirror. The σ_3 eigenstates are supposed to represent which arm of the interferometer the particle is in. The upper arm is the the σ_3 =+1 eigenstate, and the lower arm the σ_3 =-1 eigenstate. The two half silvered mirror each impliment the unitary matrix $U = \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3)$.

In addition we will place into the upper arm between the two half silvered mirros a two level measuring apparatus, whose intial state is the -1 eigenstate of Σ_3. If the particle is in the upper arm between the two half silvered mirrors, then the state of the apparatus goes from the lowest to the upper (+1) state with amplitude $\cos(theta)$. If the particle is in the lower branch, the apparatus remains in thelower (-1) state.

After the measurement, the particle goes through another half silvered mirror.

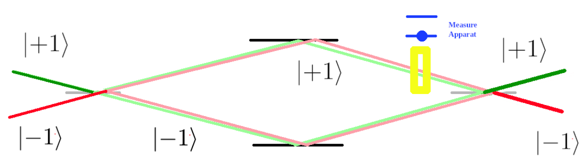

Figure 1: The interferometer with upper path defined as the +1 eigenstate of σ_3 and -1 the lower. The second diffraction grating (grey) "mirror" may or may not be in place. The red trajectory is the particle coming in from the lower ($|-1\rangle_\sigma$) while the green comes in from above. ($|1\rangle_\sigma$). The light green and light red are superpositions an the two tracks. The diagram is if the second half silvered mirror is in place. Otherwise the light tracks would continue through. The red and green tracks have been displaced horizontally from each other just to make them visible. In actuallity, they would follow the same paths between the mirrors. The yellow box is where the interaction between the apparatus and the particle takes place, while the blue is the apparatus in its initial state of of Σ_3 having value of -1.

a)σ_3 is the operator which says whether or not the particle is in the upper or lower region. Ie, if σ_3 is measured, if the answer is +1, it is upper, if -1 it is lower. Assuming that the initial state of the particle is $|-1\rangle_{\sigma_3}$. What is the state of the particle just after it has gone through the first half-silvered mirror.

$\frac{1}{\sqrt{2}}(\sigma_1+\sigma_3)\left|-\right\rangle = \frac{1}{\sqrt{2}}(\left|-\right\rangle - \left|+\right\rangle)a$
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b) What is the state of the whole system after the particle has passed the measuring apparatus?
Show that

$$U_M = \frac{1}{2}(\sin(\theta) I_\Sigma + i\ \cos(\theta)\Sigma_2)(I_\sigma+\sigma_3) + I_\Sigma(I_\sigma - \sigma_3) \tag{1}$$

is a Unitary matrix which impliments the above measurement protocol. (prove that this is a Unitary matrix, and that it produces the needed result) I_Σ and I_σ are the unit matrices for the apparatus and particle respectively.

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$$U_M^\dagger U_M = \frac{1}{2}(\sin(\theta) I_\Sigma - i\cos(\theta)\Sigma_2)(I_\sigma+\sigma_3) + I_\Sigma(I_\sigma-\sigma_3) \tag{2}$$

$$I_\Sigma^2 = I_\Sigma;\quad I_\Sigma\Sigma_i = \Sigma_i;\quad \Sigma_2^2 = I_\Sigma \tag{3}$$

$$(I_\sigma \pm \sigma_3)^2 = 2(I_\sigma \pm \sigma_3 \tag{4}$$

$$(I_\sigma+\sigma_3)(I_\sigma-\sigma_3) = 0 \tag{5}$$

$$(I_\sigma-\sigma_3)(I_\sigma+\sigma_3) = 0 \tag{6}$$

$$U_M^\dagger U_M = \frac{1}{4}(sin(\theta) I_\Sigma + cos(\theta)\Sigma_2)^2(I_\sigma+\sigma_3) + I_\Sigma(I_\sigma-\sigma_3) \tag{7}$$

$$= I_\Sigma I_\sigma \tag{8}$$

Then

$$U_M(\frac{1}{\sqrt{2}}\left|\downarrow\right\rangle[\left|-\right\rangle + \left|+\right\rangle) = \frac{1}{2\sqrt{2}}(\sin(\theta)\left|\downarrow\right\rangle - \cos(\theta)\left|\uparrow\right\rangle]2\left|+\right\rangle\left|\downarrow\right\rangle\left|-\right\rangle) \tag{9}$$

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c) If the second half silvered mirror is in place, what is the state of the whole system after that half silvered mirror?

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WE now multiply the result by the half silvered mirror Unitary matrix $I_\Sigma(\frac{1}{\sqrt{2}}(\sigma_z+\sigma_x)$ to give

$$\frac{1}{2}[(\sin(\theta)(\left|\downarrow\right\rangle) - \cos(\theta)(\left|\uparrow\right\rangle](\left|+\right\rangle + \left|-\right\rangle)(\left|\downarrow\right\rangle(\left|-\right\rangle - \left|+\right\rangle)) \tag{10}$$

$$= \left|\downarrow\right\rangle[(sin(\theta)-1)\left|+\right\rangle(sin(\theta)+1)\left|-\right\rangle) - \cos(\theta)(\left|\uparrow\right\rangle((\left|+\right\rangle + \left|-\right\rangle)) \tag{11}$$

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d)If the second half-silverd mirror is in place, and the Σ_3 attribute is of the apparatus is measured and found to have value +1. what is the probability that the particle will have been found to come out in the upper state?

After the measurement of Σ_3, the state is $|\uparrow\rangle$ and the state of particle is $(sin(\theta)-1)\,|+\rangle + (sin(\theta)-1)\,|-\rangle$ and the probability of comint out in the up direction is $\frac{(sin(\theta)-1)^2}{(sin(\theta)-1)^2+(sin(\theta)+1)^2)} = \frac{(sin(\theta)+1)^2}{sin(\theta)^2+1}$ If $\theta = 0$, this is 0, while if $\theta = \pi/2$, this is 1/2.

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e) What if instead the Σ_1 attribute of the apparatus is measured and found to have value +1. What will the probability be that the particle comes out the upper state from the second mirror?

The +1 eigenstate of Σ_1 is $\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ and so, if we call the final state of the system plus apparatus Ψ, then the (unnormalised(state of the particle will be

$$\frac{1}{\sqrt{2}}(\langle\uparrow| + \langle\downarrow|)\,|\Psi\rangle = \frac{1}{\sqrt{2}}[(sin(\theta)-1)\,|+\rangle\,(sin(\theta)+1)\,|-\rangle) - \cos(\theta)((|+\rangle + |-\rangle))] \quad (12)$$

$$= \frac{1}{\sqrt{2}}[(\sin(\theta)-1-cos(\theta))\,|+\rangle + (\sin(\theta)+1-cos(\theta))\,|-\rangle)] \quad (13)$$

If $\theta = \pi/2$, this is proportional to $|+\rangle$ Ie, one gets interference, so the particle always comes out of just one of the ports.

Ie, the conditional measurement of the apparatus changes the interference of the particle .

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Does it matter if the Σ_1 is measured before or after the particle measurement after it has exited the apparatus?for the answer to the last question?

No

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f) What is the reduced density matrix for the particle after the measuring apparatus but before the second half silvered mirror?

$$Tr_\Sigma\left(\frac{1}{2\sqrt{2}}(\sin(\theta)\,|\downarrow\rangle - \cos(\theta)\,|\uparrow\rangle]2\,|+\rangle\,|\downarrow\rangle\,|-\rangle)\frac{1}{2\sqrt{2}}(\sin(\theta)\,\langle\downarrow| - \cos(\theta)\,\langle\uparrow|]2\,\langle+|\,|\downarrow\rangle\,\langle-|)\right) \quad (14)$$

$$= \frac{1}{8}Tr_\Sigma\,([|\downarrow\rangle\,(2\sin(\theta)\,|+\rangle + |-\rangle) + |\uparrow\rangle\,(-2\cos(\theta)\,|+\rangle)][\langle\downarrow|\,(2\sin(\theta)\,\langle+| + \langle-|) + |\uparrow\rangle\,(-2\cos(\theta)\,\langle+|)]) \quad (15)$$

$$= \frac{1}{8}\,((2\sin(\theta)\,|+\rangle + |-\rangle)(2\sin(\theta)\,\langle+| + \langle-|) + (-2\cos(\theta)\,|+\rangle)(-2\cos(\theta)\,\langle+|)) \quad (16)$$

It is not diagonal.

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g) What is the reduced density matrix for the measuring apparatus after the measurement interaction but before the half silved mirror?

2)Lets say we have a the state

$$|\psi\rangle = sin(\theta)\,|+,\uparrow\rangle + cos(\theta)(sin(\phi)\,|-,\uparrow\rangle + cos(\phi)\,|-,\downarrow\rangle) \tag{17}$$

In the lecture I chose $sin(\phi) = sin(\theta)/cos(\theta)$.

As in the lecture A and B are operators on the first system (its Hilbert space is spanned by $|+\rangle, |-\rangle$ while C and D are of the second spanned by $|\uparrow\rangle, |\downarrow\rangle$. Lets say that $|\pm\rangle$ are the eigenstates of A.

a) What is the reduced density matrix for the first and second systems.

$$\begin{aligned}
\rho_1 &= Tr_\sigma(sin(\theta)\,|+,\uparrow\rangle + cos(\theta)(sin(\phi)\,|-,\uparrow\rangle + cos(\phi)\,|-,\downarrow\rangle)) && (18)\\
&\qquad \otimes(sin(\theta)\,\langle +,\uparrow| + cos(\theta)(sin(\phi)\,\langle -,\uparrow| + cos(\phi)\,\langle -,\downarrow|)) && (19)\\
&= \sin^2(\theta)\,|\uparrow\rangle\langle\uparrow| + \cos(\theta)^2(sin(\phi)\,|\uparrow\rangle + cos(\phi)\,|\downarrow\rangle)(sin(\phi)\,\langle\uparrow| + cos(\phi)\,\langle\downarrow|) && (20)\\
&= (sin^2(\theta) + cos^2(\theta)sin^2(\phi))\,|\uparrow\rangle\langle\uparrow| + cos^2(\theta)cos^2(\phi)\,|\downarrow\rangle\langle\downarrow| && (21)\\
&\quad +cos^2(\theta)cos(\phi)sin(\phi)(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|) && (22)
\end{aligned}$$

The trace is 1 as it should be.

For the second system we trace out over the Σ eigenvalues,

$$\begin{aligned}
\rho_2 &= Tr_\Sigma(sin(\theta)\,|+,\uparrow\rangle + cos(\theta)(sin(\phi)\,|-,\uparrow\rangle + cos(\phi)\,|-,\downarrow\rangle)) && (23)\\
&\quad \otimes(sin(\theta)\,\langle +,\uparrow| + cos(\theta)(sin(\phi)\,\langle -,\uparrow| + cos(\phi)\,\langle -,\downarrow|)) && (24)\\
&= (\sin(\theta)\,|+\rangle + cos(\theta)(sin(\phi)\,|-\rangle)(\sin(\theta)\,\langle +| + cos(\theta)(sin(\phi)\,\langle -|) + \cos(\theta)^2\cos(\phi)^2\,|-\rangle\langle -| && (25)\\
&= \sin(\theta)^2\,|+\rangle\langle +| + +cos(\theta)^2\,|-\rangle\langle -|\,i && (26)\\
&\quad +(\sin(\theta)\cos(\theta)\sin(\phi))(|+\rangle\langle -| + |-\rangle\langle +|) && (27)
\end{aligned}$$

which again has unit trace, and is not diagonal. The determinant, which is the same in both cases is

$$det(\rho_2) = \sin(\theta)^2\cos(\theta)^2(1 - \sin(\phi)^2) = \sin(\theta)^2\cos(\theta)^2 cos(\phi)^2 \tag{28}$$

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b) What are the +1 eigenvectors For $A,\ B,\ C,\ D$ which make up the Hardy chain? Ie, if A is measured to have value +1, then C has value +1, If C has value +1, then B has value +1. If B has value +1 then D has value +1.

If $A \rightarrow 1$ then its state is $|+\rangle$. This multiplies the vector $\sin(\theta)\,|\uparrow\rangle$ which normalized is $|\uparrow\rangle$ which is the +1 eigenvalue of Σ_3.

$$\langle\uparrow|\,|\psi\rangle = sin(\theta)\,|+\rangle + \cos(\theta)\sin(\phi)\,|-\rangle \tag{29}$$

which normalised is

$$|B \to 1\rangle = \frac{\sin(\theta)\,|+\rangle + \cos(\theta)\sin(\phi)\,|-\rangle}{\sqrt{sin(\theta)^2 + \cos(\theta)^2 \sin(\phi)^2}}$$

This is the +1 eigenvector of B.

Finally the +1 eigenvector of D is

$$|D \to 1\rangle = N_D \langle B \to 1|\,|\psi\rangle = N_D[(\sin(\theta)^2 + \cos(\theta)^2 \sin(\phi)^2)\,|\uparrow\rangle + cos(\theta)^2 \cos(\phi)\sin(\phi)\,|\downarrow\rangle] \tag{30}$$

where N has value

$$N_D = \frac{1}{\sqrt{(\sin(\theta)^2 + \cos(\theta)^2 \sin(\phi)^2)^2 + (\cos(\theta)^2 \sin(\phi)^2)^2}} \tag{31}$$

If A is measured and has value +1, the the probability that D has value 1 is

$$|\,\langle\uparrow|\,|D \to 1\rangle\,|^2 = \frac{(\sin(\theta)^2 + cos(\theta)^2 sin(\phi)^2}{(\sin(\theta)^2 + \cos(\theta)^2 \sin(\phi)^2)^2 + (\cos(\theta)^2 \sin(\phi)^2)^2} \tag{32}$$

$$= \frac{= sin(\phi)^2(1 - sin(\theta)^2) + sin(\theta)^2}{sin(\phi)^2(1 - sin(\theta)^4) + sin(\theta)^4} \tag{33}$$

Note that this is minimized by taking $\sin(\phi) = \pm\frac{sin(\theta)}{\sqrt{1+\sin(\theta)^2}}$ giving the probability of

$$\mathcal{P} = 4\frac{sin(\theta)^2}{(1 + sin(\theta)^2)^2} \tag{34}$$

This gives roughly the same probability as the assumption I made in class that $\sin(\phi) = \tan(\theta)$ for small θ, the class assumption does not work for $\theta > \pi/4$ since that would require $sin(\phi) > 1$.

Note (You are not expected to have given the argument below) that this also implies that this Hardy chain could be used for a completely arbitrary state. If the state was

Let us assume that the state is

$$|\psi\rangle = \alpha\,|+\rangle\,|\uparrow\rangle + \beta\,|+\rangle\,|\downarrow\rangle + \gamma\,|-\rangle\,|\uparrow\rangle + \delta\,|-\rangle\,|\downarrow\rangle \tag{35}$$

$$\sin(\theta)^2 = \alpha^2 + \beta^2 \tag{36}$$

$$|\tilde{\uparrow}\rangle = \frac{1}{|\sin\theta)|}(\alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle) \tag{37}$$

$$|\tilde{\downarrow}\rangle = \frac{1}{|\sin\theta)|}(\alpha\,|\downarrow\rangle - \beta\,|\uparrow\rangle) \tag{38}$$

$$\sin(\phi) = \frac{1}{\sin(\theta)\cos(\theta)}\alpha\gamma + \beta\delta \tag{39}$$

we have

$$\psi = \sin(\theta)\left|+,\tilde{\uparrow}\right\rangle + \cos(\theta)\left|-\right\rangle(\sin(\phi)\left|\tilde{\uparrow}\right\rangle + \cos(\phi)\left|\tilde{\downarrow}\right\rangle \tag{40}$$

I.e., the Hardy chain can be applied to any arbitrary two-2level system.
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c) What is the probability that, if A has value +1, then D has value +1?

If A has value 1, then the state of the the second system is 1, with vector $|\uparrow\rangle$. The Probability that D has value 1 is then

$$\mathcal{P} = \frac{((\sin(\theta)^2 + \cos(\theta)^2\sin(\phi)^2))^2}{(\sin(\theta)^2 + \cos(\theta)^2\sin(\phi)^2))^2 + cos(\theta)^4 cos(\phi)^2\sin(\phi)^2} \tag{41}$$

3) No Cloning

Argue that there exists no single unitiary matrix which will transform $(\alpha|+\rangle + \beta|-\rangle)|\downarrow\rangle$ to $(\alpha|+\rangle + \beta|-\rangle)((\alpha|\uparrow\rangle + \beta|\downarrow\rangle))$ for arbitrary (normalized) values of β, α Ie, you cannot transform a clone a generic state. Why does problem 4 not fall afoul of this theorem?

Any Unitary transformation is linear. But the transformed vector is a function of $\alpha^2,\ \beta^2,\ \alpha\beta$ which are all nonlinear function of α and β. Thus no unitary transformation can behave as required.
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4). Bell states

Given two 2-dimension systems, with basis states

$$|B_1\rangle = \frac{1}{\sqrt{2}}(|+\rangle|\uparrow\rangle + |-\rangle|\downarrow\rangle) \tag{42}$$

$$|B_2\rangle = \frac{1}{\sqrt{2}}(|+\rangle|\uparrow\rangle - |-\rangle|\downarrow\rangle) \tag{43}$$

$$|B_3\rangle = \frac{1}{\sqrt{2}}(|+\rangle|\downarrow\rangle + |-\rangle|\uparrow\rangle) \tag{44}$$

$$|B_4\rangle = \frac{1}{\sqrt{2}}(|+\rangle|\downarrow\rangle - |-\rangle|\uparrow\rangle) \tag{45}$$

Show that these are eigenstates of the operator

$$S = \sigma_3\Sigma_3 + 2\sigma_1\Sigma_1 \tag{46}$$

where σ_i operate on the $|+\rangle,\ |-\rangle$ subspace and $Sigma_i$ operate on the other subspace, and $|\pm\rangle$ are the eigenstates of the σ_3 and $|\updownarrow\rangle$ of the $\Sigma 3$.

$$\sigma_3\Sigma_3 |+,\uparrow\rangle = |+,\uparrow\rangle \quad (47)$$
$$\sigma_3\Sigma_3 |-,\uparrow\rangle = -|-,\uparrow\rangle \quad (48)$$
$$\sigma_3\Sigma_3 |+,\downarrow\rangle = -|+,\downarrow\rangle \quad (49)$$
$$\sigma_3\Sigma_3 |-,\downarrow\rangle = |-,\downarrow\rangle \quad (50)$$
$$\sigma_1\Sigma_1 |+,\uparrow\rangle = |-,\downarrow\rangle \quad (51)$$
$$\sigma_1\Sigma_1 |-,\uparrow\rangle = |+,\downarrow\rangle \quad (52)$$
$$\sigma_1\Sigma_1 |+,\downarrow\rangle = |-,\uparrow\rangle \quad (53)$$
$$\sigma_1\Sigma_1 |-,\downarrow\rangle = |+,\uparrow\rangle \quad (54)$$
$$(55)$$

Thus

$$(\sigma_3\Sigma_3 + 2\sigma_1\Sigma_1)|B_1\rangle = +|B1\rangle + 2|B1\rangle = 3|B1\rangle \quad (56)$$
$$(\sigma_3\Sigma_3 + 2\sigma_1\Sigma_1)|B_2\rangle = +|B2\rangle - 2|B2\rangle = -1|B2\rangle \quad (57)$$
$$(\sigma_3\Sigma_3 + 2\sigma_1\Sigma_1)|B_3\rangle = -|B3\rangle + 2|B3\rangle = |B3\rangle \quad (58)$$
$$(\sigma_3\Sigma_3 + 2\sigma_1\Sigma_1)|B_4\rangle = -|B4\rangle - 2|B4\rangle = -3|B4\rangle \quad (59)$$
$$(60)$$

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What is the reduced density matrix for the σ system for each of these states.

In each case it is $\frac{1}{2}$ the identity matrix

$$\rho_1 = \frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|) \quad (61)$$

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Pauli Matrices

$$\sigma_1 |+1\rangle = |-1\rangle; \qquad \sigma_1 |-1\rangle = |+1\rangle \quad (62)$$
$$\sigma_2 |+1\rangle = i|-1\rangle; \qquad \sigma_2 |-1\rangle = -i|+1\rangle \quad (63)$$
$$\sigma_1 |-1\rangle = -|-1\rangle; \qquad \sigma_1 |+1\rangle = |+1\rangle \quad (64)$$