Physics 501-20

Norm vs Frequency

It is generally true that the signs of the norm and the frequency are the same– ie, a mode with time dependence $e^{-i\omega t}$; $\omega > 0$ also has positive norm. But there are situations in which this is not true. The most trivial is if the Hamiltonian is negative.

$$H = -\frac{1}{2}(p^2 + \omega q^2) \tag{1}$$

The solutions again have time dependence $e^{\pm i\omega t}$, but in this case , from the equation of motion, $\partial_t q = -p$, we have $p = i\omega q$ for the $e^{-i\omega t}$ mode, and the norm

$$< q, q >= \frac{i}{2}(q^* p - p^* q) = -\omega|q|^2 \tag{2}$$

is negative. Ie, the $e^{-i\omega t}$ has negative norm, and must be associated with the annihilation operator in order that $[A, A^\dagger] = 1$ comes from $[Q, P] = i$.

Another example, which is closer to situations one has in physical systems, is the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2) + v(p_1 q_2 - p_2 q_1) \tag{3}$$

The easiest way to solve this is to note that the first term is symmetric under the rotation of q1 to q2 and p1 to p2 and vice versa. The equations of motion are

$$\dot{q}_1 = p_1 + vq_2 \qquad \dot{p}_1 = -q1 + vp_2 \tag{4}$$
$$\dot{q}_2 = p_2 - vq_1 \qquad \dot{p}_2 = -q2 - vp_1 \tag{5}$$
$$\tag{6}$$

The eigenvalues are $\omega = [1 + v, -(1 + v), 1 - v. - (1 - v)]$ and the modes have (taking q_1 real)

$$\omega = 1 + v; \qquad iq_2 = -p_2 = ip_1 = q1 \tag{7}$$
$$\omega = -(1 + v) \qquad -iq_2 = -p_2 = -ip_1 = q1 \tag{8}$$
$$\omega = (1 - v) \qquad -iq_2 = p_2 = ip_1 = q1 \tag{9}$$
$$\omega = -(1 - v) \qquad iq_2 = p_2 = -ip_1 = q1 \tag{10}$$
$$\tag{11}$$

The second and fourth are complex conjugates of the first and third nodes. (if the Hamiltonian has $\Omega^2(q_1^2 + q_2^2)$ instead of $(q_1^2 + q_2^2)$then the ω will be $\pm(\Omega \pm v)$ and the the solutions will have $p_1, p_1 \to \sqrt{\Omega}p_1, \sqrt{\Omega}p_2$ and $q_1, q_2 \to q_1/\sqrt{\Omega}, q_2/\sqrt{\Omega})$

That these are solutions, can most easily be seen by substituting into the equations of motion.

If $1 > v > 0$ then the first and third cases are positive norm, and the 2nd and fourth, complex conjugates of the first and third, and are negative norm. However, if $v > 1$, then while the third now has negative frequency, it still has positive norm. The norm depends only on the values of q_i, p_i, not on the frequency. Since the above values of q_i, p_i are independent of v, those norms are also independent of v. but the frequency of the third and fouth cases have changed and flipped signs.