Physics 501-20
Assignment 4

1)Consider a field ϕ like the field of sounds in a fluid with Lagrangian action

$$S = \int \left((\partial_t\phi)^2 - (F(\partial_x)\phi)^2\right) dxdt \tag{1}$$

Where F is some analytic function of the operator ∂_x (ie, we can define F by its taylor series expansion).

For example

$$\sinh(\partial_x) = \sum_{r=1}^{\infty} \frac{\partial_x^{2r-1}}{(2r-1)!} \tag{2}$$

and

$$\sinh(\partial_x)e^{ikx} = \sum_r \frac{\partial_x^{2r-1}}{(2r-1)!} e^{ikx} \tag{3}$$

$$= \sum_r \frac{(ik)^{2r-1}}{(2r-1)!} e^{ikx} = i\sin(k)e^{ikx} \tag{4}$$

Ie, $F(\partial_x)e^{ikx} = F(ik)e^{ikx}$

a)Now carry out a coordinate transformation,$y = x - vt$ and find the Lagrangian action in the new coordinates $t,\ y$ for any function F.

b) What is the momentum conjugate to the field in the $t,\ y$ coordinates.

c) What is the norm of the field in both coordinates? Ie, show, as I claimed, that the norm is the same in both coordinates even if the Hamiltonian diagonalization frequency changes in the two coordinates.

2) Consider a Harmonic oscillator

$$H = \frac{1}{2}\left(\omega(p^2 + x^2) + \tilde{\omega}(\tilde{p}^2 + \tilde{x}^2)\right) \tag{5}$$

With Annihilation operators $A,\ \tilde{A}$.

a) Show that the normalized n quantum state in each case is

$$|n\rangle = \frac{A^n}{n!} |0\rangle > \tag{6}$$

Now consider the some other annihilation operators

$$B = \alpha A + \beta \tilde{A}^\dagger \tag{7}$$

$$\tilde{B} = \gamma \tilde{A} + \delta A^\dagger \tag{8}$$

b)From the commutation relations that $B,\ B^\dagger$ must satisfy, find the relation between the coefficients $\alpha,\ \beta, \gamma,\ \delta$ that must be satisfied if B and $\tilde{B}$ are to be independent annihilation operators. Show that a solution exists if all of $\alpha,\ \beta, \gamma,\ \delta$ are real and positive.

c) What is the vacuum state of the operators $B,\ \tilde{B}$. Express them in terms of the states $|n\rangle$ and $|\tilde{n}\rangle$ of the original $A, \tilde{A}$.

d)What is the reduced density matrix of this state for the first A system. Show that this density matrix can be expressed as a thermal density matrix $\rho = Ne^{\frac{1}{2}\omega(p^2+x^2)/T}$ where N is a normalisation factor. (Show that $(\frac{1}{2}\omega(p^2 + x^2)\,|n\rangle = (n+\frac{1}{2})\omega\,|n\rangle$. Show that $|n\rangle$ is an eigenstate of ρ with eigenvalue $\lambda(n)$. What is $\lambda(n)$? What is N?

Recall that Maxwell showed that, in thermal equilibrium, if the energy of a state is E, then the probability of that state is proportional to e^{E/k_BT}.