Physics 501-22
Assignment 5

1)Consider a field ϕ like the field of sounds in a fluid with Lagrangian action

$$S = \int \left((\partial_t\phi)^2 - (F(\partial_x)\phi)^2\right) dxdt \tag{1}$$

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(There should have been a $\frac{1}{2}$ in front of this! Oh well, I will use it as it stands)

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Where F is some analytic function of the operator ∂_x (ie, we can define F by its taylor series expansion).

For example

$$\sinh(\partial_x) = \sum_{r=1}^{\infty} \frac{\partial_x^{2r-1}}{(2r-1)!} \tag{2}$$

and

$$\sinh(\partial_x)e^{ikx} = \sum_r \frac{\partial_x^{2r-1}}{(2r-1)!}e^{ikx} \tag{3}$$

$$= \sum_r \frac{(ik)^{2r-1}}{(2r-1)!}e^{ikx} = i\sin(k)e^{ikx} \tag{4}$$

Ie, $F(\partial_x)e^{ikx} = F(ik)e^{ikx}$

a)Now carry out a coordinate transformation,$y = x - vt$ and find the Lagrangian action in the new coordinates $t,\ y$ for any function F.

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Assume that one has a solution $\phi(t,x)$ of the equation. Then the new solution is $\phi(t,y+vt)$. The $\partial_t(\phi(t,x)) = \partial_t\phi(t,y+vt) - v\partial_y\phi(t,y+vt)$ and $\partial_x(\phi(t,x)) = \partial_y(\phi(t,y+vt))$. Thus the equation for in the new coordinates is

$$\partial_t^2(\phi(t,x) - F^2(\partial_x)\phi(t,x) = (\partial_t + v\partial_y)(\partial_t + v\partial_y))\phi(t,y+vt) - F(-\partial_y)F(\partial_y)\phi(t,y+vt) \tag{5}$$

(The $F(-\partial_y)$ comes from integration by parts of the operator $F(\partial_y$ since every intgration by parts reverses the sign of ∂_y.

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b) What is the momentum conjugate to the field in the $t,\ y$ coordinates.

—————————————— Using the same argument and defining $\hat\phi(t,y) = \phi(t,x+vt)$ we get the Lagrangian

$$S = \int \left((\partial_t\phi)^2 - (F(\partial_x)\phi)^2\right) dxdt \tag{6}$$

$$= \int \left(((\partial_t + v\partial_y)\hat\phi(t,y))^2 - (F(\partial_y)\hat\phi(t,y))^2\right) dydt \tag{7}$$

The conjugate momentum is $\frac{\delta S}{\delta(\partial_t \hat{\phi}(t,y)}$ which in this case is $\hat{\pi} = 2((\partial_t + v\partial_y)\hat{\phi})$

c) What is the norm of the field in both coordinates? Ie, show, as I claimed, that the norm is the same in both coordinates even if the Hamiltonian diagonalization frequency changes in the two coordinates.

The norm is $\int \phi^*(t,x)\pi(t,x) - \pi(t,x)^*\phi(t,x)dx$ and $\int \hat{\phi}^*(t,y)\hat{\pi}(t,y) - \hat{\pi}(t,y)^*\hat{\phi}(t,y)dy$ Writing the momentum in terms of derivatives of the field we see

$$\int \hat{\phi}^*(t,y)\hat{\pi}(t,y) - \hat{\pi}(t,y)^*\hat{\phi}(t,y)dy = 2\int \phi^*(t,y+vt)(\partial_t - v\partial_y)\phi(t,y+vt) - (\partial_t - v\partial_y)\phi^*(t,y+vt)\phi(t,y$$

$$= 2\int \phi(t,x)^*\partial_t\phi(t,x) - \partial_t\phi(t,x)^*\phi$$

$$= \int \phi(t,x)^*\pi(t,x) = \pi(t,x)\phi$$

as required

2) Consider a Harmonic oscillator

$$H = \frac{1}{2}\left(\omega(p^2 + x^2) + \tilde{\omega}(\tilde{p}^2 + \tilde{x}^2)\right) \tag{11}$$

With Annihilation operators $A,\ \tilde{A}$.

a) Show that the normalized n quantum state in each case is

$$|n\rangle = \frac{A^n}{n!}|0\rangle > \tag{12}$$

—————————————- (And now two mistakes in one equation. This should be $\frac{A^{\dagger n}}{\sqrt{n!}}|0\rangle >$

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$$[A, A^\dagger] = 1 \tag{13}$$

$$(A\dagger^n|0\rangle)^\dagger A\dagger^n|0\rangle = \langle 0|A^nA^{\dagger n}|0\rangle \tag{14}$$

$$= \langle 0|(A^{n-1}[A, A^{\dagger n}] + A^{\dagger n}A)|0\rangle \tag{15}$$

$$= \langle 0|(A^{n-1}(\sum_r A^{\dagger r}[A, A^\dagger]A^{\dagger(n-r-1)} + 0)|0\rangle \tag{16}$$

$$= \langle 0|(nA^{n-1}A^{\dagger(n-1)}|0\rangle \tag{17}$$

$$= \langle 0|(n(n-1)....(1))|0\rangle = n! \tag{18}$$

Thus to normalise the state we must divide by the square root of this.

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Now consider the some other annihilation operators

$$B = \alpha A + \beta\tilde{A}^\dagger \tag{19}$$

$$\tilde{B} = \gamma\tilde{A} + \delta A^\dagger \tag{20}$$

b)From the commutation relations that $B,\ B^\dagger$ must satisfy, find the relation between the coefficients $\alpha,\ \beta,\gamma,\ \delta$ that must be satisfied if B and $\tilde{B}$ are to be independent annihilation operators. Show that a solution exists if all of $\alpha,\ \beta,\gamma,\ \delta$ are real and positive.

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We want

$$[B, B^dagger] = [tildeB, tildeB^dagger] = 1 \tag{21}$$
$$[B, tildeB] = [B, tildeB^dagger] = 0 \tag{22}$$

from $[A, A^\dagger] = [\tilde{A}, \tilde{A}^\dagger] = 1$ and$[A, \tilde{A}] = [A, \tilde{A}^\dagger] = 0$ We thus get the 4 equation

$$[\alpha A + \beta\tilde{A}^\dagger, \alpha A^\dagger + \beta A] = \alpha^2 - \beta^2 = 1 \tag{23}$$
$$[\gamma\tilde{A} + \delta A^\dagger, \gamma\tilde{A}^\dagger + \delta A] = \gamma^2 - \delta^2 = 1 \tag{24}$$
$$[\alpha A + \beta\tilde{A}^\dagger, \gamma\tilde{A} + \delta A^\dagger] = \alpha\delta - \beta\gamma = 0 \tag{25}$$
$$[\alpha A + \beta\tilde{A}^\dagger, \gamma\tilde{A}^\dagger + \delta A] = 0 \tag{26}$$

The first says that $\alpha = \cosh(\phi),\ \beta = \sin(\phi)$ for some ϕ. The second says similarly that $\gamma = \cosh(\psi),\ \delta = \sin(\psi)$ for some ψ. Thus the third says that $\cosh(\phi)\sinh(\psi) = \cosh(\psi)\sin(\phi)$ or $\tanh(phi) = \tanh(\psi)$, which implies that $\phi = \psi$.

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c) What is the vacuum state of the operators $B,\ \tilde{B}$. Express them in terms of the states $|n\rangle$ and $|\tilde{n}\rangle$ of the original $A, \tilde{A}$.

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$$B\,|0_{B\tilde{B}}\rangle = \tilde{B}\,|0_{B\tilde{B}}\rangle \tag{27}$$

This gives

$$(cosh(\phi)A + \sinh(\phi)\tilde{A}^\dagger)\,|0_{B\tilde{B}}\rangle = (cosh(\phi)\tilde{A} + \sinh(\phi)A^\dagger)\,|0_{B\tilde{B}}\rangle = 0 \tag{28}$$

Again, we assume that $|0_{B\tilde{B}}\rangle = F(A^\dagger, \tilde{A}^\dagger)\,|0\rangle$ and that $A = \partial_{A^\dagger},\quad \tilde{A} = \partial_{\tilde{A}^\dagger}$ to give

$$\cosh(\phi)\partial_{A^\dagger}F(A^\dagger, \tilde{A}^\dagger) + \sinh(\phi)\tilde{A}^\dagger F(A^\dagger, \tilde{A}^\dagger = 0 \tag{29}$$
$$\cosh(\phi)\partial_{\tilde{A}^\dagger}F(A^\dagger, \tilde{A}^\dagger) + \sinh(\phi)A^\dagger F(A^\dagger, \tilde{A}^\dagger = 0 \tag{30}$$

which gives

$$F = Ne^{(-\tanh(\phi)A^\dagger\tilde{A}^\dagger)} \tag{31}$$

where N is a normalisation factor.

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d) What is the reduced density matrix of this state for the first A system. Show that this density matrix can be expressed as a thermal density matrix

$\rho = Ne^{\frac{1}{2}\omega(p^2+x^2)/T}$ where N is a normalisation factor. (Show that $(\frac{1}{2}\omega(p^2 + x^2)\left|n\right\rangle = (n+\frac{1}{2})\omega\left|n\right\rangle$. Show that $\left|n\right\rangle$ is an eigenstate of ρ with eigenvalue $\lambda(n)$. What is $\lambda(n)$? What is N?

Recall that Maxwell showed that, in thermal equilibrium, if the energy of a state is E, then the probability of that state is proportional to e^{E/k_BT}.

$$Ne^{-\tanh(\phi)A^\dagger\tilde{A}^\dagger}\left|0\right\rangle = N\sum_n \frac{1}{n!}(-\tanh(\theta))^n A^{\dagger n}\tilde{A}^{\dagger n}\left|0\right\rangle \tag{32}$$

$$= N(-\tanh(\phi))^n \left|n,n\right\rangle \tag{33}$$

Thus the reduced density matrix is

$$\phi_R = N^2\sum_n \left\langle n\right|\left|n\right\rangle\left|n\right\rangle |\tanh(\phi)|^{2n}\left\langle n\right| = \sum_n e^{2\ln(|\tanh\phi)|}\left|n\right\rangle\left\langle n\right| \tag{34}$$

If we write

$$2ln(|\tanh(\phi)) = -\omega/(k_BT) \tag{35}$$

where ω is the frequency of the oscillator, then the density matrix is

$$\rho_R = N^2\sum_n (e^{-n\omega/(k_BT)}\left|n\right\rangle\left\langle n\right| \tag{36}$$

which is just the Maxwell equilibrium state of a quantum harmonic oscillator of frequency ω and temperature T. Note that $Tr(\rho_R) = 1$ which gives

$$N^2\sum_n e^{-2nln(|\tanh(\phi)|} = \frac{1}{1-e^{-2ln(|\tanh(\phi)|}} \tag{37}$$

$$N = \sqrt{1-e^{2ln(|\tanh(\phi)|}} \tag{38}$$

Note that since $|tanh(x)| < 1$ the argument to the exponential is always negative.