Physics 501-22
Assignment 4

1) Consider the field theory with Lagrangian

$$\mathcal{L} = \int (\partial_t\phi - v\partial_x\phi)^2 - (\partial_x\phi)^2 - (\partial_x^2\phi)^2 dx \tag{1}$$

where x represents the one dimensional coordinate.

i)Find the Hamiltonian.

$$\pi = \frac{\delta\mathcal{L}}{\delta\partial_t\phi} = \partial_t\phi - v\partial_x\phi \tag{2}$$

$$H = \int \pi\partial_t\phi - \mathcal{L} = \int \pi(\pi + v\partial_x\phi) - \frac{1}{2}(\pi^2 - (\partial_x\phi)^2 - (\partial_x^2\phi)^2)dx \tag{3}$$

$$= \frac{1}{2}\int (\pi^2 + 2\pi v\partial_x + (\partial_x\phi)^2 + (\partial_x^2\phi)^2)dx \tag{4}$$

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ii) Assume $v > 1$. Find the modes which diagonalize this Hamiltonian. (Hint use the spatial Fourier transform), and assume that the temporal solution goes as $e^-i\omega t$. What are the solutions for ω as a function of k the wavenumber? What is the condition on ω that the solutions have positive norm?

choose $\phi = \phi_k(t)e^{ikx}$ and $\pi_k(t)e^{ikx}$ to give

$$\partial_t\phi_k = \pi_k + ik\phi_k \tag{5}$$

$$\partial_t\pi_k = ik\pi_k + k^2\phi_k + k^4\phi_k \tag{6}$$

Now chose $\partial_t\phi_k = -i\omega\phi_k$ and similarly for π. This gives

$$(\omega - vk)^2 = k^2 + k^4 \tag{7}$$

or

$$\omega = vk \pm k\sqrt{1+k^2} \tag{8}$$

if $v >> 1$, then for posive k, both solutions for ω are greater than zero as k increases from 0, until finally one of the solutions goes negative. Thus for small positive k, there are no negative ω.

But the norm is

$$i(\phi_k^*\pi_k - \pi_k^*\phi_k) = i(\phi_k^*(-i\omega - ivk)\phi_k + (i\omega + vk)\phi_k^*\phi_k \tag{9}$$

$$= \pm k\sqrt{1+k^2} \tag{10}$$

Ie, a negative norm solution can have positive ω and vice versa.

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This is the field theory which represents sound waves in a BEC, where the spatial coordinates are expressed in terms of the "healing length" and the time is chosen so that the velocity of sound at long wavelengths is unity. This is looking at the BEC in a frame moving with velocity v with respect to the rest frame of the BEC.

2) Consider the harmonic oscillator with the Hamiltonian

$$H = \frac{1}{2}(p^2 + \mu(t)(px + xp)) \tag{11}$$

with .

a) What are the equations of motion for $p,\ x$.

$$\partial_t x = p + m(t)x; \qquad \partial_t p = -m(t)p \tag{12}$$

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b) Carry out the first stage of the adiabatic transformation. What equation must mu satisfy in order that the transformed equation are those of a free particle?

$$S = p\partial_t x - \frac{1}{2}(p + m(t)x)^2 - m^2(t)x^2) \tag{13}$$

$$\hat{p} = p + m(t)x; \qquad \hat{x} = x \tag{14}$$

$$S = (\hat{p} - m(t)x)\partial_t x - \frac{1}{2}(\hat{p}^2 - m^2 x^2 \tag{15}$$

$$= \hat{p}\partial_t x + \frac{1}{2}\partial_t m\hat{x}^2 - \frac{1}{2}(\hat{p}^2 - m^2\hat{x}^2 \tag{16}$$

$$= \hat{p}\partial_t x - \frac{1}{2}(\hat{p}^2 - (m^2 + \partial_t m)\hat{x}^2 \tag{17}$$

In order that the $\hat{x}^2$ term vanish, we need

$$\partial_t m + m^2 = 0 \tag{18}$$

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3. Consider the the field theory given by the one spatial dimensional Hamiltonian in a one dimensional cosmology writen in terms of the modes

$$\mathcal{L} = \frac{1}{2}\int \left(a(t)\left((\partial_t\phi_k(t))^2 - \frac{k^2}{a(t)^2}\phi_k(t)^2 \right)\right) dk \tag{19}$$

a) What is the Hamiltonian for this system?

$$\pi = a(t)\partial_t\phi_k \tag{20}$$

$$H = \frac{1}{2}(\frac{\pi_k^2}{a(t)} + \frac{k^2}{a(t)}\phi_k^2) \tag{21}$$

$$= \frac{1}{2a(t)}(\pi_k^2 + k^2\phi_k)^2 \tag{22}$$

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b)What is the "particle creation" at time $t+\delta$ if we use the Hamiltonan diagonalisation to define particles?a Is it finite?

If we redefine $\tau = \int \frac{dt}{a(t)}$, then the terms are all constant coefficients, and if we define the modes by Hamiltonian diagonalisation over τ there will be no particle creation.

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c) What is the first order adiabatic transformation of this system?

See above. Alol you need to do is redefine the time, and you have a system with constant coefficients.

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