Physics 501-22
Cosmology

We will now look at the Genral Relativity which gives the solution, but the main result is that the universe expands as a function of time. In particular the distance between two nearby objects increases, not because they are moving but because new space is created beteen the objects. If we use x to label the position of ojects at rest, then the distance function between nearby objects is given by (Pythagoras's theorem)

$$ds^2_{space} = a(t)^2(dx^2 + dy^2 + dz^2) = a^2(d\vec{x} \cdot d\vec{x}). \tag{1}$$

Ie, the distance between nearby objects increases as $a(t)$. The special relativitistic spacetime distance is given by

$$ds^2 = dt^2 - ds^2_{space} = dt^2 - a(t)^2(d\vec{x} \cdot d\vec{x}) \tag{2}$$

The equation of motion of a scalar field is

$$\frac{1}{a^3}\partial_t a^3 \partial_t \phi - \frac{1}{a^2}\nabla^2\phi = 0 \tag{3}$$

which can be derived from a Lagrangian

$$\mathcal{L} = \frac{1}{2}\int a^3(\partial_t\phi^2 - \frac{1}{a(t)^2}(\nabla\phi \cdot \nabla\phi))d^3x \tag{4}$$

$a(t)^3d^3x$ is the volume element of space, given that the spatial distances increase as $a(t)dx$. This is like $rd\theta$ where a little change in the coordinate θ corresponds to an actual physical distance of $rd\theta$.

The conjugate momentum to $\phi(t, x)$ is o

$$\frac{\delta\mathcal{L}}{\delta\partial_t\phi(t,x)} = \pi \tag{5}$$

or

$$\pi = a^3\partial_t\phi \tag{6}$$

and the Hamiltonian is

$$H = \int \pi\partial_t\phi d^3x - \mathcal{L} = \frac{1}{2}(\frac{\pi^2}{a^3} + a(t)|\nabla\phi|^2) \tag{7}$$

The Hamiltonian action is

$$S = \int \pi\partial_t\phi d^3x - H = \int \left[\pi\partial_t\phi - \frac{1}{2}\int(\frac{\pi^2}{a(t)^3} + a(t)\nabla\phi\cdot\nabla\phi\right] d^3x \tag{8}$$

The equations of motion are

$$\partial_t\phi = \frac{\pi}{a^3} \tag{9}$$

$$\partial_t\pi = a\nabla^2\phi \tag{10}$$

We can write the spatial part of this in terms of exponentials of spatial coordinates $\frac{e^{ik\cdot x}}{\sqrt{(2\pi)^3}}$

$$\phi_k(t,x) = \frac{1}{\sqrt{2\pi}^3}\phi_k(t)\frac{e^{i(k\cdot x)}}{\sqrt{(2\pi)^3}}d^3x \tag{11}$$

(Note that in some places k will represent a three dimensional vector, while in others it will be length of that vector. I hope this is clear from context) and similarly for π_k, with the time dependent equations

$$\partial_t\phi_k = \frac{\pi_k}{a^3} \tag{12}$$

$$\partial_t\pi_k = ak^2\phi_k(t) \tag{13}$$

which come from a Hamilatonian action

$$\frac{H_k = \frac{1}{2}(\pi_k^2}{a^3 + ak^2\phi_k^2)} \tag{14}$$

(Just to be clear, we note that ϕ_k and π_k are actually complex. One should actually write the Hamiltonian in terms of the real and imaginary parts of ϕ_k, π_k, or use the modes $cos(k\cdot x)$, $sin(k\cdot x$.This complicates the notation, hiding the essentials of the procedure, so, at the expense of the possibility of confusion when you think of this more deeply, I will be sloppy).

Thus we have the action for each $\vec{k}$,

$$S_k = \int \pi_k(\partial_t\phi_k) - \frac{1}{2}(\frac{\pi_k^2}{a^3} + ak^2\phi_k^2)d^3kdt \tag{15}$$

$$= \int \pi_k(\partial_t\phi_k) - \frac{1}{2}\frac{k}{a}(\frac{\pi^2}{ka^2} + ka^2\phi_k^2)dtd^3k \tag{16}$$

Comparing this for each k to the expression for the adiabatic expansion we find that

$$\tau_k = \int \frac{k}{a}dt \tag{17}$$

$$\Omega_k = ka^2 \tag{18}$$

We thus have

$$\hat{\pi}_k = \frac{\pi_k}{\sqrt{k}a} - \frac{\dot{a}}{a}\sqrt{k}a\phi_k \tag{19}$$

$$\hat{\phi}_k = \phi_k\sqrt{k}a \tag{20}$$

where $\dot{} = \frac{d}{d\tau_k}$ and

$$\hat{H}_k = \frac{1}{2}(\hat{\pi}_k^2 + \hat{\phi}_k^2(1 - \frac{\ddot{a}}{a})) \tag{21}$$

Now this τ_k depends on k and scales as k for large k. so $\ddot{a}/a$ will scale as $\frac{1}{k^2}$ and becomes very small for large k. On the other hand for small k this will be very large, and if $\ddot{a} > 0$,

$1-\ddot{a}/a$ will go negative. In that case the solution to the equations of motion will grow or decrease exponentially, with faster growth for smaller k in terms of τ_k.

The other important relation is between the ˆ momentum and configurtion and the original.

$$\hat{\pi}_k = \pi_k/(\sqrt{k}a) + \dot{a}\frac{\phi_k}{\sqrt{k}} \tag{22}$$

$$\hat{\phi}_k = \sqrt{k}a\phi_k \tag{23}$$

Let us assume that we are looking at large enough k that that the k dependence in $\hat{H}$ can be neglected. Then the solution for $\hat{\phi}$, $\hat{\pi}$ is

$$\hat{\phi}_k = \hat{\phi}_k(0)\cos(k\hat{\tau}) + \hat{\pi}_k(0)\sin(k\hat{\tau}) \tag{24}$$

$$\hat{\pi}_k = \hat{\pi}_k(0)\cos(k\hat{\tau}) + \hat{\phi}_k(0)\sin(k\hat{\tau}) \tag{25}$$

From the equation $d\tau_k = kdt/a(t)$, we must have τ_k be proportional to k. If $a(t(\tau_k))$ is exponential, then $\ddot{a}/a$ is constant, and one can find the solution. In that case

$$d\tau_k = k\frac{dt}{a(t)} \tag{26}$$

$$dt = \frac{1}{k}a(\tau_k)d\tau_k = \alpha e^{\beta\tau_k}d\tau_k = d(\frac{\alpha}{\beta k}e^{\beta\tau_k}) \tag{27}$$

for both $a(t) = \alpha e^{\beta\tau_k}$ and t to be independent of k, we must have βk be a constant. and thus τ_k to be proportional to k. which means that $\tau_k = ln(t)\beta$ or $a(t) \propto t$.

For small k, the Lagrangian equations of motion are

$$\partial_t^2\phi_k + 3\frac{\partial_t a(t)}{a(t)}\partial_t\phi_k + \frac{k^2}{a(t)^2}\phi_k \tag{28}$$

If $k^2 << 3/2\partial_t a(t)^2$, the second term will dominate over the first, and ϕ_k will have two solutions, one constant, and the other proprotional to $a(t)^{3/2}$.

Quantization:

To quantize the field, we need to decide what the positive norm modes are that we are going to use for the definition of quanta or particles for the system. The obvious one is to use Hamiltonian diagonalization for each of the modes. However in general the Hamiltonian diagonalisation at some time t does not evolve into Hamiltonian diagonalisation at a different time. Ie, if we choose Hamiltonian diagonalisation as our definion, the evolution of the modes will take the vacuum state at one time into a non-vacuum state at a different time. This is not surprizing as the time dependence of the universe might be expected to create particles. Teh question is how many?

The Hamiltonian diagonalisation is defined by

$$\partial_t\phi_{Hk}(t) \rightarrow -i\omega(t)\phi_{Hk}(t) \tag{29}$$

$$\partial_t\pi_{Hk}(t) \rightarrow -i\omega(t)\pi_H k(t) \tag{30}$$

The equation of motion is

$$\partial_t\phi_k(t) = \frac{\pi_k(t)}{a(t)^3} \tag{31}$$

$$\partial_t\pi_k(t) = -k^2 a(t)\phi_k \tag{32}$$

This gives

$$\omega(t)^2 = \frac{k^2}{a(t)^2} \tag{33}$$

$$N^2 = i(\phi_{HK}(t)^*(-i\omega(t)a(t)^3\phi_{Hk}(t)) - (i\omega(t)a(t)^3\phi_{Hk}(t)^*\phi_{Hk}(t)) \tag{34}$$

$$= |\phi_k(t)|^2 2\omega(t)a(t)^3 = |\phi_k(t)|^2 2ka(t)^2 \tag{35}$$

or, for N to be 1 (unit norm)

$$\phi_{Hk}(t) = \frac{1}{a(t)\sqrt{2k}} \tag{36}$$

$$\pi_{Hk}(t) = a(t)^3(-i\omega(t)\phi_k(t)) = -i\sqrt{k/2}a(t) \tag{37}$$

However, Let us choose the mode to be the Hamiltonian diagonalization moded at time t. The evolution of the mode is given by the evolution equaitons and we have

$$\phi_k(t+\delta)) = \phi_{Hk}(t) + \frac{\pi_{Hk}(t)}{a^3(t)}\delta + O(\delta^2) \tag{38}$$

$$= \frac{1}{a(t)\sqrt{2k}} - i\frac{\sqrt{k/2}}{a(t)^2}\delta + O(\delta^2) \tag{39}$$

$$\pi_k(t+\delta) = \pi_{Hk}(t) - \frac{k^2}{a(t)}\phi_{Hk}(t)\delta \tag{40}$$

$$= -i\sqrt{k/2}a(t) - k^2\frac{1}{a(t)^3\sqrt{2k}}\delta \tag{41}$$

while the Hamiltonian diagonalisation gives

$$\phi_{Hk}(t+\delta) = (1-\mathcal{H}\delta)\frac{1}{a(t)\sqrt{2k}} \tag{42}$$

$$\pi_{H,k}(t+\delta) = (1+\mathcal{H}\delta)_(- ia(t)\sqrt{k/2}) \tag{43}$$

where $\mathcal{H} = \frac{\partial_t a(t)}{a(t)}$ is called the Hubble constant.

We now take the inner product of this evolution with the complex conjucate of the Hamiltonian diagonalization at $t+\delta$ to find out how much negative norm (with respect to the Hamiltonian diagonalisation) that the evolution has produced.

$$\begin{aligned}
\beta &= < \phi_{Hk}(t+\delta)^*, \phi_k(t+\delta) > && (44)\\
&= i\left(\phi_{Hk}(t+\delta)\pi_k(t+\delta) - \pi_{Hk}(t+\delta)\phi_k(t+\delta)\right) && (45)\\
&= 2\mathcal{H}\delta && (46)
\end{aligned}$$

We can define two sets of annihilation operators, $A_{Hk} =< \phi_{Hk}, \Phi >$, which uses the Hamiltonian diagonalisation modes at each time t. Since the diagonalisation modes do not obey the equations of motion, $A_{H,k}$ depend on time. The others are $A_k =< \phi_k, \Phi >$. Since $\phi_k(t+\delta)$ obey the equations of motion in δ, the associated annihilation operators is time independent. Taking the intial conditions that at t, $\phi_k(t) = \phi)Hk(t)$ and $\pi_k(t) = \pi_{Hk}(t)$, and

$$\phi_k(t+\delta) = \phi_{Hk}(t)\pi_k(t+\delta) = \phi_{Hk}(t) \tag{47}$$

The annihilation operator

$$A_{Hk}(t+\delta) =< \phi_{Hk}(t+\delta), A_k(t+\delta)\phi_k(t,\delta)(t+\delta) + A_k^\dagger(t+\delta)\phi_k^*(t+\delta) + \beta A_k^\dagger \tag{48}$$

$$=< \phi_k(t+\delta), \phi_{Hk}(t+\delta) > + < \phi_k^*(t+\delta), \phi_{Hk}(t+\delta) > A_{Hk}^\dagger > \tag{49}$$

Let us assume that at time t, we have the vaccum state with respect to $A_k(t)$ which is just the vacuum state with respect to $A_{Hk}(t)$. But in the Heisenberg representation, the state is a constant in time, so the state of the system satisfies $A_k(t+\delta)|0\rangle = A_{Hk}(t)|0\rangle = 0$. The number of Hamiltonian diagonalisation particles is $N_H(t+\delta) = A_{Hk}^\dagger(t+\delta)A_{Hk}(t+\delta)$ and

$$\langle 0| (alpha^* A_{Hk}^\dagger(t) + beta^* A_{kH})(\alpha A_{Hk}(t) + \beta A_{Hk}^\dagger(t))|0\rangle \tag{50}$$

$$= \||\beta|^2 \langle 0| A_{Hk}(t)A_{Hk}^\dagger(t)|0\rangle \tag{51}$$

$$= |\beta|^2 = \mathcal{H}^2\delta^2 \tag{52}$$

The effective Hamiltonian for each mode is

$$H_k = \frac{1}{2}(\frac{1}{a^3}\pi_k^2 + k^2 a\phi_k^2) \tag{53}$$

Let us make the assymptotic transformation to the ˆ variables and Hamiltonian, and time τ

$$\hat{H}_{\vec{k}} = \frac{1}{2}(\hat{\pi}_{\vec{k}}^2 + (1 - \frac{\partial_{\tau_k}^2 a(\tau_k)}{a(\tau_k)})\hat{\phi}_{\vec{k}}^2) \tag{54}$$

where

$$\tau_k = \int \frac{k}{a(t)} dt \tag{55}$$

$$a(\tau_k) = a(t(\tau_k)) \tag{56}$$

$$\hat{\phi}_{\vec{k}}(t) = \sqrt{k}a(\tau_k)\phi_{\vec{k}} \tag{57}$$

$$\hat{\pi}_{\vec{k}} = \frac{1}{\sqrt{k}a(\tau_k)}\pi_{\vec{k}} + \frac{\dot{a}}{a}\hat{\phi}_{\vec{k}} \tag{58}$$

Ie, from the previous notes $\Omega = ka^2$ and

$$H_k = \frac{1}{2}(\hat{\pi}_k^2 + (1 - \frac{\partial_{\tau_k^2} a(\tau_k)}{a(\tau_k)}\phi_k^2 \tag{59}$$

Using the Hamiltonian diagonalization, we have $a^2 \to (1 - \frac{\partial_{\tau_k}^2 a(\tau_k)}{a(\tau_k)})$ and

$$2\mathcal{H} \to 2\frac{\partial_{\tau_k}\sqrt{(1 - \frac{\partial_{\tau_k}^2 a(\tau_k)}{a(\tau_k)}}}{\sqrt{(1 - \frac{\partial_{\tau_k}^2 a(\tau_k)}{a(\tau_k)}}} = \frac{\partial_{\tau_k}\frac{\partial_{\tau_k}^2 a(\tau_k)}{a(\tau_k)}}{(1 - \frac{\partial_{\tau_k}^2 a(\tau_k)}{a(\tau_k)})} \tag{60}$$

which is β.

Now τ_k is proportional to k, and thus β will be proportional to $\frac{1}{k^3}$, and β^2 will be proportional to $\frac{1}{k^6}$. Integrating over k^2dk, for large k the upper limit in the particle creation rate ntegral will fall off $\frac{1}{K^3}$ which goes raplidly to zero. Ie, the number of particles created will be finite.

Of course $\hat{H}$ is not the "real" Hamiltonian (although surely any cannonical transformation has a valid a claim to reality as any other), or the real energy of the system.

This whole argument, which was given by L Parker (joined later by S Fulling) in the late 1960's and early 1970's raises the troublesome question– what does one mean by particles in quantum field theory in General Relativity?

It is a problem which is still troublesome even now, 50 years later.