**Supplementary comments on Twins Paradox**

The twin's paradox is the following argument: A sees B's clock as running slowly, and B sees A's clock as running slowly. Why then, if B travels out and then returns, is it B's clock, and not A's clock which is slow when they compare them.

The short answer is that it is B turns around, that changes his velocity, while A does not. [Note that special relativity does not institute complete relativity. It is not true that all motions are relative. Only motions with constant velocity are indistinguishable. Accelerations are still absolute. One can tell whether or not one is physically accelerated or not, simply by doing local experiments.]

A longer answer is to remember that when we say that A or B see the other's clock as going slower, it is with respect to the specific synchronisation of clocks which special relativity demands. When B changes his velocity, he must also change his synchronisation in order to say that A's clock runs slower than his clock does. But, when he changes synchronisation, the time he assigns to A at some specific point in A's journey also changes. The agrument below examines this in detail.

Consider B travelling out with velocity v for time T (in A's frame) and then turning around and travelling back with velocity -v. Let us call A's time and space coordinates $t, x$, B's coordinates on the outward journey $t', x'$ and on the return journey $t'', x''$. We will choose the origin of the coordinates of B on the outward journey to be such that $t = t' = x = x' = 0$ to be the start of the journey. The turning point will be assumed to be , in the $xt$ system at $t = T, x = vT$ which will have $t', x'$ coordinates of

$$t' = \gamma(T - \frac{v}{c^2}vT) = \frac{T}{\gamma} \tag{1}$$

$$x' = 0 \tag{2}$$

Let us choose the $t'', x''$ coordinates so that that turning point in the $t'', x''$ frame occurs at $t'' = \frac{T}{\gamma}, x'' = 0$. Ie, we choose the $t''x''$ coordinates so the turning point of B's journey occurs at the same time and place in the $t''x''$ system as in the $t'x'$ system. The times for B are continuous at the location of B. This will actually require a generalisation of the Lorentz transformation B in the double primed system.

$$t'' = \frac{T}{\gamma} + \gamma((t - T) + \frac{v}{c^2}(x - vT)) \tag{3}$$

$$x'' = \gamma((x - vT) + v(t - T)) \tag{4}$$

Ie, we have to shift the origin of the $t'', x''$ system so as to make sure that the time and position as seen for B is continuous. Substituting $t = T$ and $x = vT$ into these relations we find that $t'' = \frac{T}{\gamma}$ and $x'' = 0$ as required.

Now, the problem is that the intersection of the line $t' = \frac{T}{\gamma}$ with A's path ($x = 0$) and the line $\frac{t''}{\gamma}$ with A's line ($x = 0$) occur at very different times, $t$.

For $t' = \frac{T}{\gamma}$ ,the turn around time, and $x = 0$ we have $\frac{T}{\gamma} = \gamma t$ or $t = \frac{T}{\gamma^2}$ while $t'' = \frac{T}{\gamma}$, and $x = 0$ gives us $\frac{T}{\gamma} = \frac{T}{\gamma} + \gamma((t - T) + \frac{v}{c^2}(0 - vT))$ or $t = (1 + \frac{v^2}{c^2}T) = 2T - \frac{T}{\gamma^2}$.

Thus the parts of A's trajectory which correspond to $t' < \frac{T}{\gamma}$ and $t'' > \frac{T}{\gamma}$ are of length $2\frac{T}{\gamma^2} = \frac{1}{\gamma}(\frac{2T}{\gamma})$ which is shorter, by a factor of $\frac{1}{\gamma}$ than B's total time of $\frac{2T}{\gamma}$, just as the "paradox" argument would suggest. But the parts of A's curve which have $t' < \frac{T}{\gamma}$ and $t'' > \frac{T}{\gamma}$ are NOT the whole of A's curve. These miss a whole chunk of length $2T\frac{v^2}{c^2}$ precisely because of the resynchronisation which occurs for B's clocks when B turns around. That resynchronisation of B's clocks in his frame must be taken into account by B in calculating A's total time. While the switch from the $t'x'$ froma to the $t''x''$ frame is clear at B's location ($x' = x'' = 0$), it is not at all obvious at what time that switch should take place at other locations in B's frames. Certainly it cannot take place at $t' = t'' = \frac{T}{\gamma}$. Even B agrees that these two times do not correspond to the same spacetime points because of the resynchronisation of B's clocks.

The relation between the $t'x'$ and $t''x''$ frames is given by (from substituting the transformation from $tx$ to $t'x'$ into eqns 3,4)

$$\begin{aligned}
t'' &= \frac{T}{\gamma} + \gamma\left(((\gamma(t' + \frac{v}{c^2}x') - T) + \frac{v}{c^2}(\gamma(x' + vt) - vT)\right) &(5)\\
&= \gamma^2\left((1 + \frac{v^2}{c^2})t' + \frac{2v}{c^2}x'\right) + 2v^2\gamma T &(6)\\
x'' &= \gamma\left(\gamma(x' + vt') + v\gamma(t' + \frac{v}{c^2}x') - 2vT\right) &(7)\\
&= \gamma^2(1 + \frac{v^2}{c^2})x' + \gamma^2 vt' - 2v\gamma T &(8)
\end{aligned}$$

Thus the $t'' = \frac{T}{\gamma}$ line, the surface of simultenaity in the $t''x''$ frame to B's turnaround corresponds to the surface

$$(t' - \frac{T}{\gamma}) + \frac{\frac{2v}{c^2}}{1+\frac{v^2}{c^2}} x' = 0 \tag{9}$$

For negative $x'$, which is where A is located, this in in the future ($t' > \frac{T}{\gamma}$) of the turnaround simultenaity surface in the $t'x'$ frame. This, if we take the $t'' = \frac{T}{\gamma}$ this as the cut where we go from the $t'x'$ to the $t''x''$ frames, we find that there is a portion of the future (B's time to the future of the turnaround) which is doubly covered– by both the $t'x'$ coordinates and the $t''x''$ system. A's journey will look strange. During the whole of

$$t' < \frac{T}{\gamma} - \frac{\frac{2v}{c^2}}{1+\frac{v^2}{c^2}} x' \tag{10}$$

B will "see" (have reports sent back to him from his cohorts at various positions $x'$ or $x''$) A travel away from him at velocity $v$. But A's curve will extend long into the future after the turnaround time. Then suddenly, when B transfers to the $t''x''$ frame, A will instantly travel backwards in "time" to the turnaround time, and continue forward with velocity $v$ until A joins B. Along each leg of A's journey, B will see A's clock as running slower than his. But A double covers the future times. A't outward journey extend into the future of the turnaround, and then jumps back to the turnaround time. That extra length of the trip into the future in the $t'x'$ frame is just the right amount of time to make up the deficit that B calculates for A's clock running slowly.

Thus, B will say that A travels with a slow clock long beyond the $t' = \frac{T}{\gamma}$ turnaround time. Then suddenly when he resynchronises his clocks, A instantly travels backwards in time to the $t'' = \frac{T}{\gamma}$ time, and then travels forward again with a slow clock.

Clearly that jump into the apparent past is the result of B's resynchronisation of his clocks. However, he MUST resynchronise if he is to see A's clock as always being slower than his. B can only say that A's clock runs more slowly than his **if** he synchronises his clocks on each part of his journey in the special relativistic manner.

**No resynchronisation**

What if B refuses to resynchronize his clocks Ie, at $t' = \frac{T}{\gamma}$ he goes to the $t''x''$ frame but does not resynchronise his clocks. In problem 1 of assignment 2, you show that he could simply maintain his clocks on the $t'x'$

synchronisation. However in this case he does not find that A's clock runs slower than his does. In this case A's clock runs faster than his does. (Note that comparing a moving clock to the time in your frame requires calculating the time at different positions in your frame– the moving clock has moved. Thus the rate of a moving clock depends on synchronisation, just as does the length of moving rod.)

In the above I choose to resynchronze at the turnaround time at the $t''$ turnaround time. I could have done it at some other time (even the $t = T$ time). In each case the details of A's journey will change, but in each case we will find the same "jump back into the past" phenomenon in B's frame. However that choise of when to do the resynchronisation is arbitrary.

A much more straightforward way is to carry out the procedure of problem 2 of the assignment 2. In this way, you are not comparing clocks to imaginary clocks that B resynchronises throught his frame. Here the calculation of where A is and the comparison of A's time to B's time occurs directly by B calculating the location of A from the angular measurement of A's height, and B computes the time he assigns to A by using the constancy of light and the calculated location of A. Ie, it is done purely by means of a local set of measurements made by B at his location alone.

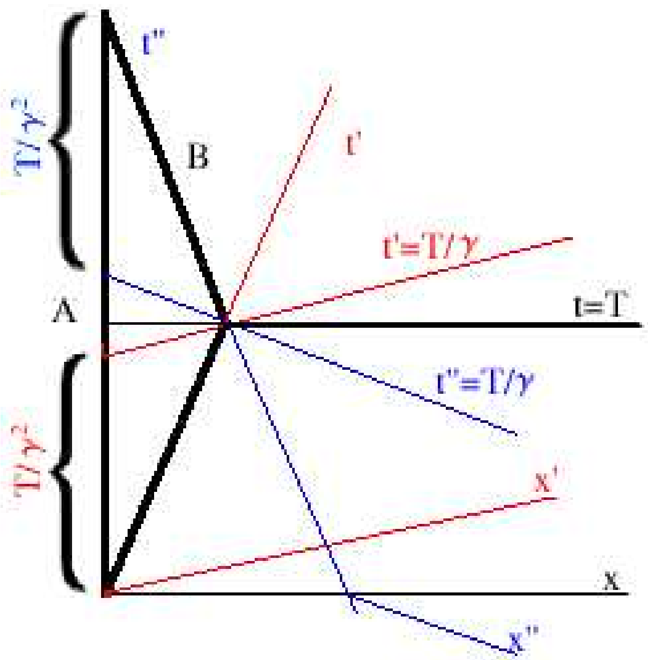


Figure 1: Diagram of the various coordinate systems in the analysis of the twin's motion. The blue are the $t''x''$ system, and the red the $t'x'$. The black is the $tx$. The thick lines are the motion of A and B as seen in the $tx$ system. Note the gap in the description of A due to the synchronization change when going from $t'x'$ to $t''x''$.

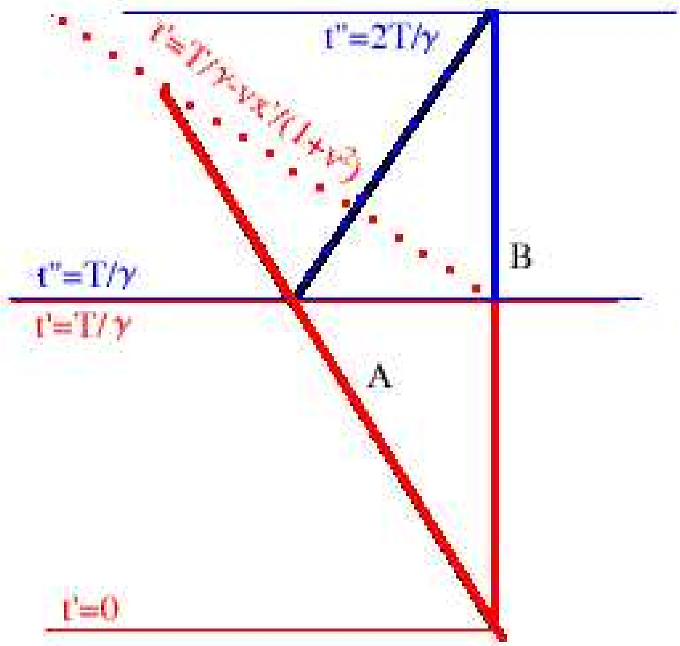


Figure 2: The motion of A as seen by B, with the $t'x'$ system extended into the future past the turnaround time so as to cover the gap in the description of A. Note that at times A has two positions at the same "time" where those times are in the $t'x'$ and $t''x''$ systems. The red portions describing A and B's paths are in the $t'x'$ system while the blue are in the $t''x''$ system.