Physics 200-04

Supplemental Problems 3

1. Show that the three Pauli sigma matrices are also all unitary matrices. How do the three sigma matrices transform under each of these three unitary transformations?

The three Pauli spin matrices are hermitean, so the test for unitarity $U^\dagger U = I$ becomes $\sigma_i^2 = I$. But we know this is true from the multiplication of the σ matrices.

$$\begin{aligned}
\sigma_1\sigma_1\sigma_1^\dagger &= \sigma_1 \\
\sigma_1\sigma_2\sigma_1^\dagger &= i\sigma_3\sigma_1 = i^2\sigma_2 = -\sigma_2 \\
\sigma_1\sigma_3\sigma_1^\dagger &= -i\sigma_2\sigma_1 = (-i)^2\sigma_3 = -\sigma_3 \\
\sigma_2\sigma_1\sigma_2^\dagger &= -\sigma_1 \\
\sigma_2\sigma_2\sigma_2^\dagger &= \sigma_2 \\
\sigma_2\sigma_3\sigma_2^\dagger &= -\sigma_3 \\
\sigma_3\sigma_1\sigma_3^\dagger &= -\sigma_1 \\
\sigma_3\sigma_2\sigma_3^\dagger &= -\sigma_2 \\
\sigma_3\sigma_3\sigma_3^\dagger &= \sigma_3
\end{aligned} \quad (1)$$

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2. What are the eigenvalues and eigenvectors of each of the following matrices?

a)

$$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \quad (2)$$

$$\begin{aligned}
&1 \pm \sqrt{2} \\
&\frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix} \\
&\frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix} 1+\sqrt{2} \\ -1 \end{pmatrix}
\end{aligned} \quad (3)$$

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b)

$$\frac{1}{25}\begin{pmatrix} -7 & -24i \\ 24i & 7 \end{pmatrix} \tag{4}$$

$$\begin{gathered} \pm 1 \\ \frac{1}{5}\begin{pmatrix} 3 \\ 4i \end{pmatrix} \\ \frac{1}{5}\begin{pmatrix} 4i \\ 3 \end{pmatrix} \end{gathered} \tag{5}$$

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c)

$$\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix} \tag{6}$$

$$\begin{gathered} \pm\sqrt{3} \\ \frac{1}{\sqrt{6-2\sqrt{3}}}\begin{pmatrix} 1+i \\ \sqrt{3}-1 \end{pmatrix} \\ \frac{1}{\sqrt{6-2\sqrt{3}}}\begin{pmatrix} \sqrt{3}-1 \\ -1+i \end{pmatrix} \end{gathered} \tag{7}$$

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3) A two level system is found to have the lower eigenvalue of the physical attribute represented by

$$\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix}$$

For each of the attributes represented by the matrices of problem 2, what is the probability that it will be found to have the upper eigenvalue of that attribute?

The eigenvalues are $\pm\sqrt{3}$. The eigenvector corresponding to the negative (lower) eigenvalue is

$$|\psi\rangle = \frac{1}{\sqrt{6-2\sqrt{3}}}\begin{pmatrix} \sqrt{3}-1 \\ -1+i \end{pmatrix} \tag{8}$$

The probabilities that one is in the upper eigenstate for each of the attributes of problem 2 is

$$\frac{1}{(4+2\sqrt{(2)})(6-2\sqrt{(3)})}|1(\sqrt{(3)}-1)+(1+\sqrt{(2)})(-1+i)|^2$$
$$\frac{1}{(4+2\sqrt{(2)})(6-2\sqrt{(3)})}\left(-2+\sqrt{(3)}-\sqrt{(2)}+i(1+\sqrt{(2)})\right)\left(-2+\sqrt{(3)}-\sqrt{(2)}-i(1+\sqrt{(2)})\right)$$
$$\frac{1}{(4+2\sqrt{(2)})(6-2\sqrt{(3)})}\left((-2+\sqrt{(3)}-\sqrt{(2)})^2+(1+\sqrt{(2)^2})\right)$$
$$=1/2$$

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4) The Hamiltonian is given by

$$H=\hbar\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

The system is found at time t=0 to have the value +1 for the attribute represented by σ_2. A sequence of measurements of σ_2 is performed at $t=1$ and $t=2$. What is the probability that both of these measurements will give a value of +1?

The positive eigenstate of σ_2 is $|\psi\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$ The Schroedinger equations of motion are

$$i\hbar\frac{d|\psi\rangle}{dt}=H|\psi\rangle$$
$$i\hbar\frac{d\psi_1}{dt}=\hbar\psi_1$$
$$i\hbar\frac{d\psi_2}{dt}=0 \qquad (14)$$

Thus

$$\begin{aligned} \psi_1(t) &= e^{-it}\psi_1(0) \\ \psi_2(t) &= \psi_2(0) \end{aligned} \tag{15}$$

Between time t=0 and 1, the state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-it} \\ i \end{pmatrix} \tag{16}$$

and the probability of finding the system in the upper eigenstate is

$$Prob = \left|\frac{1}{2}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} e^{-it} \\ i \end{pmatrix}\right|^2 = \frac{1}{4}(2 + 2cos(t)) = .7702 \tag{17}$$

After the measurement the system is again in the state $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}$ and the time development proceeds in the same way. Again the probability is the same .7702. the joint probability is then $.7702^2 = .5931$.

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5) Show that

$$[A, BC] = [A, B]C + B[A, C] \tag{18}$$

$$\begin{aligned} [A, BC] &= ABC - BCA \\ [A, B]C + B[A, C] &= (AB - BA)C + B(AC - CA) = ABC - BAC + BAC - BCA = ABC - \end{aligned}$$

Assuming that the position operator and momentum operator obey

$$[X, P] = i\hbar I \tag{20}$$

show that

$$[X^n, P] = i\hbar n X^{n-1} \tag{21}$$

$$[X^n, P] = [X^{n-1}, P]X + X^{n-1}[X, P] = [X^{n-1}, P]X + i\hbar X^{n-1} \tag{22}$$

working through we find an extra such term for each position, finally giving us

$$[X^n, P] = ni\hbar X^{n-1} = i\hbar\frac{dX^n}{dX} \tag{23}$$

Thus for any power series

$$[V(X), P] = i\hbar \frac{dV(X)}{dX} \tag{24}$$

If the potential $V(x)$ is a function of of x which can be writen as a power series, argue that the Heisenberg equation of motion for P, if the Hamiltonian is

$$H = \frac{1}{2m}P^2 + V(X) \tag{25}$$

then

$$\frac{dP}{dt} = -V'(X) \tag{26}$$

$$i\hbar\frac{dP}{dt} = [P, H] = -[H, P] = -\frac{1}{2m}[P^2, P] - [V(X), P] = -i\hbar\frac{dV(X)}{dX} \tag{27}$$

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where $V'(x)$ is the function of x defined by $\frac{dV(x)}{dx}$, and where any power servies function

$$f(x) = \sum_n a_n x^n$$

that the equivalent function of the matrix X is defined as

$$f(X) = \sum_n a_n X^n \tag{28}$$

Note that this means that the Heisenberg equations of motion for the operators P and X look just like the classical equations of motion for the variables p and x.