Physics 200-04

Supplementary Questions 1

1. Show that in two successive Lorentz transformations in the x direction, the rapidities add (The rapidity is the argument to the cosh and sinh functions in the Lorentz transformation.)

The first transformation is

$$ct' = cosh(\theta_1)ct + sinh(\theta_1)x \tag{1}$$
$$x' = cosh(\theta_1)x + sinh(\theta_1)ct \tag{2}$$

while the second going from the prime to double prime is

$$ct'' = cosh(\theta_2)ct' + sinh(\theta_2)x' \tag{3}$$
$$x'' = cosh(\theta_2)x' + sinh(\theta_2)ct' \tag{4}$$

Plugging in from the first set to the second we find the final double primed coordinates in terms of the unprimed

$$ct'' = cosh(\theta_2)(cosh(\theta_1)ct + sinh(\theta_1)x) \tag{5}$$
$$+sinh(\theta_2)(cosh(\theta_1)x + sinh(\theta_1)ct) \tag{6}$$
$$x'' = cosh(\theta_2)(cosh(\theta_1)x + sinh(\theta_1)ct) \tag{7}$$
$$+sinh(\theta_2)(cosh(\theta_1)ct + sinh(\theta_1)x) \tag{8}$$

or

$$ct'' = (cosh(\theta_2)cosh(\theta_1) + sinh(\theta_2)sinh(\theta_1))\, ct \tag{9}$$
$$+ (cosh(\theta_2)sinh(\theta_1) + sinh(\theta_2)cosh(\theta_1))\, x \tag{10}$$
$$x'' = (cosh(\theta_2)cosh(\theta_1) + sinh(\theta_2)sinh(\theta_1))\, x \tag{11}$$
$$+ (cosh(\theta_2)sinh(\theta_1) + sinh(\theta_2)cosh(\theta_1))\, ct \tag{12}$$

But from the first assignment we have that

$$cosh(\theta_2)cosh(\theta_1) + sinh(\theta_2)sinh(\theta_1) = cosh(\theta_1 + \theta_2) \tag{13}$$
$$cosh(\theta_2)sinh(\theta_1) + sinh(\theta_2)cosh(\theta_1) = sinh(\theta_1 + \theta_2) \tag{14}$$

Thus

$$ct'' = cosh(\theta_1 + \theta_2)ct + sinh(\theta_1 + \theta_2)x \tag{15}$$
$$x'' = cosh(\theta_1 + \theta_2)x + sinh(\theta_1 + \theta_2)ct \tag{16}$$

Ie, the Lorentx transformation which is the combination of two lorentz transformations in the same direction has a rapidity (the argument of the hyperbolic functions) which is the sum of the two rapidities.

To find the velocity transformation, recall that $\tanh(\theta) = \frac{v}{c}$.
Thus

$$v_{12} = ctanh(\theta_1 + \theta_2) = c\frac{sinh(\theta_1 + \theta_2)}{cosh(\theta_1 + \theta_2)} \tag{17}$$

$$= c\frac{cosh(\theta_2)sinh(\theta_1) + sinh(\theta_2)cosh(\theta_1)}{cosh(\theta_2)cosh(\theta_1) + sinh(\theta_2)sinh(\theta_1)} \tag{18}$$

$$\tag{19}$$

Dividing both top and bottom by $cosh(\theta_2)cosh(\theta_1)$, we get

$$v_{12} = c\frac{\frac{sinh(\theta_1)}{cosh(\theta_1)} + \frac{sinh(\theta_2)}{cosh(\theta_2)}}{(1 + \frac{sinh(\theta_1)}{cosh(\theta_1)}\frac{sinh(\theta_2)}{cosh(\theta_2)}} \tag{20}$$

$$= c\frac{\frac{v_1}{c} + \frac{v_2}{c}}{1 + \frac{v_1}{c}\frac{v_2}{c}} \tag{21}$$

$$= \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} \tag{22}$$

Note that this is exactly the same as if we used the "addition of velocities" formula going from the velocity of the double primed frame in the primed frame to its velocity in the unprimed frame.

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2. What is the the time error caused by the special relativistic time delay for the GPS sattelites per orbit/per day? The sattelite obital radius is 26,500 km. (Assume that the orbital time for an orbit with radius 6500km is 90 min). Since light travels at approximately 1m/3ns, what error in distance would this time delay cause if it were not corrected for by the GPS transmitter/receiver?

We need to find the velocity of these satellites. The easiest way is to recall that the centripidal acceleration, $\frac{mv^2}{r}$ equals the gravitational force $\frac{GMm}{r^2}$ from which $v^2 \propto \frac{1}{r}$. The velocity in the orbit at 6500 km is $\frac{2\pi 6500000m}{90\ 60sec} = 7.6\ 10^3 m/sec$ and thus $v^2 \approx 5.710^7 m^2/sec^2$. Since v^2 scales as $1/r$, at $26500km$ we would have

$$v_{GPS}^2 = 5,7\ 10^8 * (6500/26500) = 1.4\ 10^7 m^2/sec^2 \tag{23}$$

and finally

$$\frac{v_{GPS}^2}{c^2} = 1.4\ 10^{-10} \tag{24}$$

Thus the gamma factor is $\gamma = \sqrt{1 - v^2/c^2} \approx 1 - \frac{1}{2}v^2/c^2$. The difference in time therefor between the frame of the earth and the sattelite is approximately $\frac{1}{2}1.4\ 10^{-7}t$. The period for one orbit is about $46100sec$ so on one orbit this amounts to 3.6 10^{-5} sec. Since light travels at 3 $10^8 m/sec$ this would amount to about 1800 meters error on one orbit. After a day (86400 sec) this would amount to 6.710^-5 sec, or about 3400m error in the locations on the earth.

3. What is the time synchronisation difference at 1000 km for two observers on the opposite sides of the earth at the equator? At the moon (400,000 km)? (Assume that the moon is just rising/setting for the two observers).

The earth rotates as one rotation per day. Since the radius of the earth is 6500km, the velocity at the surface of the earth at the equator is $2\pi 6500000/(24\,60\,60) = 470 m/sec.$Thus the difference in velocity of the two observers is $940 m/sec.$ in making the transformation between the two, the times change as $t = \gamma(t' + vx/c^2)$ Thus the time synchronization difference is $\gamma vL/c^2$ where L is the distance.

To lowest order in v/c, (since $v/c \approx 1.6\,10^{-6}$ this is just $vL/c^2 = 1.6\,10^{-6}L/3\,10^8 = 510^{-15}Lsec.$ If L is 1000km=10^6m, this time synchronisation difference is just $5\ 10^{-9}$ sec. Even at the distance to the moon (400000km), it is just $2\ 10^{-6}$ sec. Ie, the difference in time synchronisation due to the rotation of the earth is in almost all cases negligible.

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4. The velocity of light in a refractive medium at rest is c/n. What is the velocity of light in the medium as seen by an observer for whom the water moves with velocity v? (Note the relevance to the Fizeau experiment).

This is just the addition of velocities. In the frame of the water, the velocity of light in both directions is $\pm c/n$. If we go to the frame moving with velocity v, the addition of velocities says

$$w_x = \frac{\pm\frac{c}{n} - v}{1 - \frac{\pm cv}{nc^2}} \tag{25}$$

$$\approx \pm\frac{c}{n} - v + \frac{v}{n^2} \tag{26}$$

But this was exactly the equation that Fresnel derived in his silly partial ether drag theory, which Fizeau verified. Thus special relativity (with no aether or aether drag) gives just the same equation as Fresnel derived as necessary (and due to aether drag) in order that the aberration of starlight be consistant with the wave theory of light.