Physics 200-06
Midterm Exam
Oct 20 2006

This exam consists of five (5) questions. All problems are worth the same number of marks.

Note that after you receive back your marked exams, you will be allowed one week to redo the exam as an assignment. The mark you get for the midterm will be the average of the two marks (the midterm proper and the midterm done as an assignment) but in any case you cannot get less than the midterm mark.

1. Given that the Matricees A and B are given by

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 1 & -2 & 0 \end{pmatrix} \tag{1}$$

$$B = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \tag{2}$$

which of the following exist? If they exist, what are their values?

i) AB

[.5] Not defined. The number of columns of A are not the same as the number of rows of B

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ii)AB^T

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[1]

Defines since the number of col. of A (4) equals the number or rows of B (4). The result has 4 rows (number of rows of A)

$$AB^T == \begin{pmatrix} 1 & 3 & -1 \\ 0 & 1 & 0 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -1 \end{pmatrix} \tag{3}$$

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b)Recall that a Lorentz transformation matrix is defined by the relation

$$L^T G L = G \tag{4}$$

where G is defined by (assuming we are using units such that $c = 1$)

$$G = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

Show that the matrix

$$L = \begin{pmatrix} \frac{5}{3} & \frac{4}{3} & 0 & 0 \\ \frac{16}{15} & \frac{4}{3} & -\frac{3}{5} & 0 \\ \frac{4}{5} & 1 & \frac{4}{5} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

is a Lorentz transformation. Is this a boost Transformation?

[2.5]

$$L^T G = \begin{pmatrix} \frac{5}{3} & \frac{16}{15} & \frac{4}{5} & 0 \\ \frac{4}{3} & \frac{4}{3} & 1 & 0 \\ 0 & -\frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{5}{3} & \frac{16}{15} & \frac{4}{5} & 0 \\ -\frac{4}{3} & \frac{4}{3} & 1 & 0 \\ 0 & -\frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

$$(L^T G)L = \begin{pmatrix} -\frac{5}{3} & \frac{16}{15} & \frac{4}{5} & 0 \\ -\frac{4}{3} & \frac{4}{3} & 1 & 0 \\ 0 & -\frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{5}{3} & \frac{4}{3} & 0 & 0 \\ \frac{16}{15} & \frac{4}{3} & -\frac{3}{5} & 0 \\ \frac{4}{5} & 1 & \frac{4}{5} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8}$$

$$= \begin{pmatrix} -\frac{5}{3}\frac{5}{3} + \frac{16}{15}\frac{16}{15} + \frac{4}{5}\frac{4}{5} & -\frac{5}{3}\frac{4}{3} + \frac{16}{15}\frac{4}{3} + \frac{4}{5}1 & 0 + \frac{16}{15}\frac{-3}{5} + \frac{4}{5}\frac{4}{5} + 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = G \tag{9}$$

[1] The matrix is not symmetric and thus it is not a boost.
(In fact I designed it so that

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & -\frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{5}{3} & \frac{4}{3} & 0 & 0 \\ \frac{4}{3} & \frac{5}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10}$$

Which is a rotation (the first row and column are all 0 except the very first term), times a boost (it is symmetric)

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2) Explain the significance of the series of experiments which led Einstein to postulate Special Relativity. What where the postulates about the world he used in deriving Special Relativity?

The three crucial experiments were a) Bradley's aberation experiment which showed that the earth moved, b) Fizzeau experiment which showec that if there is an aether drag it is weird and the amount of drag depends on the index of refraction of the light, c) Michaelson Morley which showed that the aether drag must be complete since no effect of the earth's motion could be seen.

[1 mark each, but for full marks perhaps a bit of elaboration on at least one of them.]

The two postulates i) No experiment can tell the state of motion of any observer who is not acted upon by a force.

ii) The apeed of light in one frame is the same value c independent of direction or other attribute, and thus by the above must be c in all frames.

[2] ******************************

3) Derive the length contraction– Ie a moving meterstick appears to be shorter than the same meter stick at rest. How fast would the meter stick have to travel so that a meter stick looked like a yard stick? (a metre is almost exactly $\frac{13}{12}$ of a yard.)

Many ways of doing this. Easiest is to say in the ˜ frame, the rod is at rest with one end at $\tilde{x}_{left} = 0$ and the other at $\tilde{x}_{right} = L$. We can write this in terms of the coordinates in the other frame, as

$$\tilde{x}_{left} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(x_{left} - vt) \tag{11}$$

$$\tilde{x}_{right} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(x_{left} - vt) \tag{12}$$

Subtracting these and demanding that we measure x_{left} and x_{right} at the same time t we have

$$L = \tilde{x}_{right} - \tilde{x}_{right} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(x_{right} - x_{left}) \tag{13}$$

or

$$(x_{right} - x_{left}) = \sqrt{1-\frac{v^2}{c^2}}L \tag{14}$$

Ie the difference between the left and right coordinates of the ends at the same time t is $\sqrt{1-\frac{v^2}{c^2}}$ times the length of the rod at rest.

[1] for knowing what the length contraction is. [1] for knowing that the meadurement in the frame in which the rod is moving must be made at the same time in that frame. [3] for giving a coherent derivation of the contraction.

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4) i) Given that the force on a particle in the rest frame of the particle is of the form

$$\bar{F} = \begin{pmatrix} 0 \\ F_x \\ 0 \\ 0 \end{pmatrix} \tag{15}$$

What is the force in a frame moving with velocity v in the +x direction? What is the magnitude of the spatial components of the force in this new frame?

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[2]This is a simple transformation of the vector components into the new Lorentz frame.

$$\tilde{F}^t = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(F_t - \frac{v}{c}F_x) = \frac{\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}F_x$$
$$\tilde{F}^t = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(F_x - \frac{v}{c}F_t) = \frac{\frac{1}{c}}{\sqrt{1-\frac{v^2}{c^2}}}F_x \tag{16}$$

Thus the spatial component of the force increases when looked at in a moving frame. (a frame in which the velocity of the particle is non-zero)
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ii) Joe argues that obviously things can go faster than light. Imagine applying a force to a particle to make it go at 4/5 c. Now apply that same force again to the particle and it will again increase its velocity by 4/5c bringing the total to 8/5 c which is larger than c. What (if anything) is the problem with his argument?

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[3]There are many ways that the argument is wrong and his argument is ambiguous. If he means that in each case the force IN TH FRAME of the particle is applied, than in each case in the frame in which the particle is at rest before the application, the increase in velocity will be the same. But the change in velocity as seen in the initial frame where the particle was at rest is not simple the addition of velocities– in relativity the combination of velocities is not simple addition, but is given by the rule

$$\frac{v_f = \frac{4}{5}c + \frac{4}{5}c}{1 + \frac{4}{5}\frac{4}{5|} = \frac{\frac{8}{5}}{\frac{41}{25}}c = \frac{40}{41}c < c} \tag{17}$$

If he means that he is going to give the same force to the particle in his own frame, then from part a) the force in the rest frame of the particle just before he gives the second push is smaller by a factor of $\sqrt{1-(\frac{4}{5})^2} = \frac{3}{5}$, and thus the increase in velocity will be even less than it was in the first assumption, and the combined velocity will be even less than the about $\frac{40}{41}$
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5) A particle with velocity $\frac{4}{5}c$ and rest mass energy of 1MeV decays into two gamma rays, with one of the gamma rays travelling in the same direction as the original particle was travelling. What is the total energy of the original particle? What is the final energy of that gamma ray which travels in the same direction as the original particle did? (Note that a gamma ray is a massless particle).

Let us assume that the original particle is travelling in the direction x.
Then

$$\bar{p}_M = \begin{pmatrix} \frac{E}{c} \\ p \\ 0 \\ 0 \end{pmatrix} = \frac{E}{c}\begin{pmatrix} 1 \\ \frac{v}{c} \\ 0 \\ 0 \end{pmatrix} = \frac{Mc}{\sqrt{1-\frac{v^2}{c^2}}}\begin{pmatrix} 1 \\ \frac{v}{c} \\ 0 \\ 0 \end{pmatrix} \tag{18}$$

is the momentum before hand, with $Mc^2 = 1MeV$.

[1]The total initial energy is

$$E = \frac{Mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1MeV}{\sqrt{1-(\frac{4}{5})^2}} = \frac{5}{3}MeV \tag{19}$$

Afterwards we have two photons, with energy ϵ and ϵ' and energy momentum vectors

$$\bar{p}_\epsilon = \begin{pmatrix} \frac{\epsilon}{c} \\ \frac{\epsilon}{c} \\ 0 \\ 0 \end{pmatrix} \tag{20}$$

$$\bar{p}_{\epsilon'} = \begin{pmatrix} \frac{\epsilon'}{c} \\ \frac{-\epsilon'}{c} \\ 0 \\ 0 \end{pmatrix} \tag{21}$$

The neatest way of solving for the energy of the gamma ray is to isolate the other gamma ray's momentum and square both sides

$$\bar{P}_{\epsilon'} \cdot \bar{P}_{\epsilon'} = (\bar{P}_M - \bar{P}_\epsilon) \cdot (\bar{P}_M - \bar{P}_\epsilon) \tag{22}$$

or

$$0 = 0 - M^2c^2 - 2\bar{P}_M \cdot P_\epsilon = -M^2c^2 - 2(-\frac{\epsilon}{c}\frac{E}{c} + \frac{\epsilon}{c}\frac{E}{c}v = M^2c^2 + \frac{\epsilon}{c}\frac{E}{c}(1-\frac{v}{c}) \tag{23}$$

or

$$\epsilon = \frac{M^2c^4}{2E(1-\frac{v}{c})} = \frac{3}{2}Mc^2 = \frac{3}{2}MeV \tag{24}$$

Alternatively,

$$\begin{aligned} \epsilon + \epsilon' &= E \\ \epsilon - \epsilon' &= \frac{v}{c}E \end{aligned} \tag{25}$$

or

$$\epsilon = \frac{1}{2}\frac{5}{3}MeV(1+\frac{4}{5}) = \frac{3}{2}MeV. \tag{26}$$

[1] for conservation of energy and momentum, [1.5] for getting the right the form the vectors for the three particles, [1.5] for the solution.

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