Physics 200-04
Assignment 4

1) [Based on French 6-8]. A propulsion system has been proposed where a strong llaser is shone at a totally reflecting "sail" in space. The sail is assumed to be perfectly reflecting in its own rest frame. Ie, the energy of the photon reflected equals the incident energy in this frame (assuming that the rest mass energy of the sail is much greater than the energy of the photon) Ie, you can assume that in the frame the sail, the photon has the same energy after reflection as when it was incident.]

i) First, assume that the sail is much heavier than the particle. Show that if the sail is travelling with velocity v, the energy transfered to the sail by a single photon of incident energy ϵ (travelling in the same direction as the sail) is $2\epsilon\frac{v}{1+v}$.(This problem uses coordinates such the c=1) (Hint– transform the photon to the frame moving with the sail, assume specular reflection and then transform the reflected photon back to the original frame.)

ii) Consider the photons emitted from the source at n per second. How many photons per unit length are there travelling from the source to the sail? What is the total number between the source and the sail when the sail is a distance x from the source. How many photons per second hit the sail in the frame of the source? What is the energy transfer per unit time to the sail?

There are many ways of doing this problem. One way would be to do what I suggest in the problem. Note I will do the problem with c=1. To restore the cs, divide all velocities by c, multiply all times by c.

In the frame of the sail, the photon simply bounces off the sail, without its energy or momentum being changed since the mass of the sail is so much greater than the energy of the photon. Thus, we do a Lorentz transformation of the photon into the frame of the sail, and then take the photon that has bounced off the sail back to the frame of the observer. The photon energy momentum is

$$\begin{pmatrix} \epsilon \\ \epsilon \\ 0 \\ 0 \end{pmatrix} \tag{1}$$

Transforming to the frame of the sail travelling with velocity v in the positive direction

$$\tilde{\epsilon} = cosh(\mu)(\epsilon - v\epsilon) = \epsilon\frac{1}{\sqrt{1-v^2}}(1-v) = \epsilon\sqrt{\frac{1-v}{1+v}} \tag{2}$$

(Since we know that a light particle remains a light particle in the new frame we know that the momentum is still equal to the energy.)

After the collision the particle in the frame of the sail has energy momentum vector

$$\begin{pmatrix} \tilde{\epsilon} \\ -\tilde{\epsilon} \\ 0 \\ 0 \end{pmatrix} \tag{3}$$

since we are assuming that the mass of the sail is much much greater than ϵ. Transforming this back to the frame of the lab is equivalent to doing a Lorentz boost with velocity -v. This gives

$$\tilde{\tilde{\epsilon}} = \frac{1}{\sqrt{1-v^2}}(\tilde{\epsilon} + v(-\tilde{\epsilon}) \tag{4}$$

$$= \tilde{\epsilon}\sqrt{\frac{1-v}{1+v}}\tilde{\epsilon} = \frac{1-v}{1+v}\epsilon \tag{5}$$

Thus in the earth frame the photon now has energy momentum vector

$$\frac{1-v}{1+v}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \tag{6}$$

The change in momentum of the photon is just the change in momentum of the sail. Thus

$$\delta P_{sail} = \epsilon\left(1 + \frac{1-v}{1+v}\right) = 2\epsilon\frac{1}{1+v} \tag{7}$$

The force on the sail is just the change in momentum times the number of photons which hit the sail per second.

This problem can also be done as a "Compton effect" collision where the collision angle is 180 degrees. The conservation equation is

$$\frac{M}{\sqrt{1-v^2}}\begin{pmatrix} 1 \\ v \\ 0 \\ 0 \end{pmatrix} + \epsilon\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = M\frac{1}{\sqrt{1-\tilde{v}^2}}\begin{pmatrix} 1 \\ \tilde{v} \\ 0 \\ 0 \end{pmatrix} + \tilde{\epsilon}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \tag{8}$$

where I have explicitly written out the energy and momentum of the sail in terms of the mass and velocity. Assuming that the change in velocity is of the sail is very small, we can expand to first order in $\delta = \tilde{v} - v$.

$$\epsilon\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = M\frac{\delta}{\sqrt{1-v^2}^3}\begin{pmatrix} 1 \\ v \\ 0 \\ 0 \end{pmatrix} + M\frac{1}{\sqrt{1-v^2}}\begin{pmatrix} 0 \\ \delta \\ 0 \\ 0 \end{pmatrix} + \tilde{\epsilon}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

This is two equations for the two unknowns δ and $\tilde{\epsilon}$ with $\tilde{\epsilon} + \epsilon$ the same solution as before.

Now, to find the force on the sail if there are n photons per second sent out from the earth to the sail, we have to recall that if the sail is travelling away from us at velocity v, the number of photons hitting the sail per second (as seen by the observer on earth) is less than n. The number of photons per unit volume is $\frac{n}{c=n}$ in our units, and the volume increases as vt with time. Thus the numbr of photons "stored" (ie in transit) increases as nvt. The numbr of photons sent out goes as nt, so the number hitting the sail per unit time must be (nt-nvt)/t= n(1-v). The force on teh sail is thus $2n\epsilon\frac{1-v}{1+v}$

The energy transfer is $\epsilon - \tilde{\epsilon} = 2\epsilon\frac{v}{1+v}$ and the energy transfer per unit time is $2n\epsilon\frac{(1-v)v}{1+v}$. Restoring cs, this becomes

$$\frac{dE}{dt} = 2n\epsilon\frac{(1-\frac{v}{c})\frac{v}{c}}{1+\frac{v}{c}} \tag{10}$$

and the force is

$$F = 2n\frac{\epsilon}{c}\frac{(1-\frac{v}{c})}{1+\frac{v}{c}} \tag{11}$$

simply obtained via dimensional consideration. (Since ϵ is an energy, dividing it by c gives something with units of momentum)

[Note that calculating the force and momentum transfer was not asked for in the problem.

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2) Consider a mass, mass M, which emits a particle of mass m and leaves the large particle with mass M'. Show that for given M and M', the velocity of the mass M' is largest as m goes to zero. Ie, converting given mass to photons is the most efficient way of using that mass as a fuel for accelerating the large mass

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$$\bar{P}_M = \bar{P}_{M'} + \bar{P}_m \tag{12}$$

or

$$M\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = M'\frac{1}{\sqrt{1-V^2}}\begin{pmatrix}1\\V\\0\\0\end{pmatrix} + m\frac{1}{\sqrt{1-v^2}}\begin{pmatrix}1\\v\\0\\0\end{pmatrix} \tag{13}$$

where M and M' are given. We want to have V as a function of m the mass of the emitted particle.The easiest way to elimitate v is to take the We can rewrite this as

$$\bar{P}_m = \bar{P}_M - \bar{P}_{M'} \tag{14}$$

taking the dot product of each side with itself we have

$$-m^2 = -M^2 - M'^2 + 2(\bar{P}_M \cdot \bar{P}_{M'}) = -M^2 - M'^2 - 2MM'\frac{1}{\sqrt{1-V^2}} \tag{15}$$

Solving for V, we have

$$V^2 = 1 - \left(\frac{2MM'}{(M^2 + M'^2 - m^2)}\right)^2 \tag{16}$$

To maximize V, we want to minimize the second term on the right. Since as m increases, the denominator decreases, and thus the term increases, we want m to be as small as possible. the best would be that it be m=0. which is the light-like. Ie, the maximum velocity of the mass M' is obtained if we eject light.

3) A spaceship converts a certain fraction x of its mass into light per second (as measured by the spaceship itself) and shines the light out the back of the spaceship to accelerate it.

i) Show that the acceleration of the spaceship is constant in magnitude (ie $\bar{a}\cdot\bar{a} = |a|^2$ is constant, where $\bar{a} = \frac{d\bar{u}}{d\tau}$ is the acceleration four vector.

ii) How much of the original mass must be used up in order that the final mass of the spaceship is travelling at 0.99c

[Note: For neither of these parts do you need to solve differential equations.]

Since the acceleration is a length of a four vector, it is the same no matter what frame it is evaluated in. We can therefore evaluate it in the frame of the spaceship.

From the previous problem, for m=0, the change in velocity of the mass M at rest is

$$V = \left(1 - \left(\frac{2MM'}{M^2 + M'^2}\right)^2\right)^{\frac{1}{2}} \tag{17}$$

Since by assumption, some fraction of the mass is converted into light at each instant of length dt, we have $M' = (1 - xdt)M$ and

$$dV \quad = \quad \left(1 - \left(\frac{2(1 - xdt)}{1 + (1 - xdt)^2}\right)^2\right)^{\frac{1}{2}} \approx xdt \tag{18}$$

Thus, the change in velocity per unit time in the frame in which the vehicle is at rest is just

$$\frac{dV}{dt} = x \tag{19}$$

Now, in a frame in which the vehicle is instantaneously at rest, the proper velocity is

$$\bar{u} = \frac{1}{\sqrt{1 - V^2}} \begin{pmatrix} 1 \\ V \\ 0 \\ 0 \end{pmatrix} \tag{20}$$

and teh acceleration is

$$\frac{d\bar{u}}{d\tau} = V\frac{dV}{d\tau}\frac{1}{\sqrt{1 - V^2}^3}\begin{pmatrix} 1 \\ V \\ 0 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{1 - V^2}}\begin{pmatrix} 0 \\ \frac{dV}{d\tau} \\ 0 \\ 0 \end{pmatrix} \tag{21}$$

Since in the frame of the particle, τ is just t and V=0. we have in that frame

$$\frac{d\bar{u}}{d\tau} = \begin{pmatrix} 0 \\ \frac{dV}{d\tau} \\ 0 \\ 0 \end{pmatrix} \tag{22}$$

The length of this vector is just $\left(\frac{dV}{d\tau}\right)^2$ and since this is just x the acceleration is constant.

To find out how much of the mass the rocket must use up to get to .99c, the easiest way is to realise that we can regard this as just one big collision. The rocket ship sheds part of its mass and converts it to photons. Each photon has energy-momentum vector of the form $\epsilon \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ and the sum of all of the photons emitted by the spaceship will have exactly this same form. $E \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$.

Thus before hand we have

$$\bar{P}_M = M \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{23}$$

and afterwards we have

$$\bar{P}_{M'} = \frac{M'}{\sqrt{1-V^2}} \begin{pmatrix} 1 \\ V \\ 0 \\ 0 \end{pmatrix} \tag{24}$$

Now we are given that V is given, and we want M'. To eliminate E, the total energy of all the photons, we again write the conservation equation so that the eneggy monentum of the light is on one side of the equation and then take the dot product of both sides with themselves.

$$0 = -M^2 - M'^2 + \frac{2MM'}{\sqrt{1-V^2}} \tag{25}$$

$$M'^2 - 14.2M'M + M^2 = 0 \tag{26}$$

$$M' = M(7.1 - \sqrt{7.1^2 - 2}) \approx \frac{M}{7.1} \tag{27}$$

Ie, in order to accelerate the spaceship to .99 the speed of light, one only needs to use up about 6/7 of the mass of the spaceship as fuel.)Note that we use the negative square root in the above. the positive square root would correspond to the rocket absorbing the radiation and getting more massive.

Note that the space shuttle delivers only 1/20 of the launch weight of the vehicle to earth orbit (with a velocity of about .00002 c) – chemical rockets are a really really inefficient use of the mass of the fuel.

4) Scattering: A particle of mass m and velocity v collides with a larger mass M at rest. After the collision, the two masses are still the same, and the small mass moves at an angle θ with respect to its original motion. What is the velocity of the large mass after the collision.

Solving this is a mess. While it is perfectly possible to do so, it requires solving a quadratic equation with all coefficients an algebraic mess.

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5)Are the following possible, and if not, why not.

i) A gamma ray (which moves at the speed of light)of energy 1.5MeV decays into an electron and positron, each having the same mass, .5MeV. (Note the convention in particle physics is to measure both energies and masses in eV (electron Volts). One electron volt is 1.6 $10^{-19}J$.

No. The intial particle is a massless, null particle. When we add together the other two 4-momenta, their total Energy (over c)must be greater than their total momentum, since this is true for eachi of the particles. Thus that sum cannot have the magnitude of the momentum be equal to total energy over c.

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ii) Two gamma rays each of energy 1GeV collide to produce and electron and a positron.

The electron and positron each have a mass that is about .5MeV, so the total energy is more than enough. Furthermore, if we collide the gamma rays head on, their total momentum is zero, and by having the electron an positron fly off in opposite directions, their momentum will also add to zero.

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iii) A particle of mass M at rest collides with a gamma ray. After the collision the gamma ray is absorbed and the resultant particle still has mass M, and some non-zero velocity.

This is exactly the inverse of the problem we did in class, in the decay example. As we saw there, energy conservation demanded that the final mass be less than the initial. Here for the same reason the initial mass of he heavy particle must be less than the single final mass.

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iv) proton of Mass .9383 GeV decays into a neutron of mass .9395 GeV plus a positron of mass .5MeV and a neutrino of mass 0.

The intial mass energy is less than the total final rest mass energy. Since the products have to have some kinetic energy as well, energy conservation tells us this is impossible.

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v)One of the above neutrons decays into a proton, and electron, and a neutrino with masses as above.

In this case the sum of the rest mass energies (.9383+.0005+0=.9388GeV) of the final products is less than the intial energy (.9395GeV). Thus energetically the decay is possible. Furtehrmore, we could always have the neutrino carry zero energy and momentum and then have teh proton and the electron each carry the same momentum so that momentum conservation was obeyed as well.

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