Physics 200-05
Assignment 2

1. i) Show that

$$\sinh(\theta)\cosh(\theta') + \cosh(\theta)\sinh(\theta') = \sinh(\theta+\theta') \quad (1)$$
$$\cosh(\theta)\cosh(\theta') + \sinh(\theta)\sinh(\theta') = \cosh(\theta+\theta') \quad (2)$$

Note the similarities **and** differences with the trigonometric formulas you are (I hope) more familiar with.

$$\begin{aligned}
\frac{1}{2} & \left(e^{\theta} - e^{-\theta}\right)\frac{1}{2}\left(e^{\theta'} + e^{-\theta'}\right) + \frac{1}{2}\left(e^{\theta'} - e^{-\theta'}\right)\frac{1}{2}\left(e^{\theta} + e^{-\theta}\right) && (3)\\
= & \frac{1}{4}\left(e^{\theta+\theta'} + e^{\theta-\theta'} - e^{-\theta+\theta'} - e^{-\theta-\theta'}\right) && (4)\\
& + \frac{1}{4}\left(e^{\theta+\theta'} - e^{\theta-\theta'} + e^{-\theta+\theta'} - e^{-\theta-\theta'}\right) && (5)\\
= & \frac{1}{2}\left(e^{\theta+\theta'} - e^{-\theta-\theta'}\right) && (6)\\
= & sin(\theta+\theta') && (7)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2} & \left(e^{\theta} + e^{-\theta}\right)\frac{1}{2}\left(e^{\theta'} + e^{-\theta'}\right) + \frac{1}{2}\left(e^{\theta'} - e^{-\theta'}\right)\frac{1}{2}\left(e^{\theta} - e^{-\theta}\right) && (8)\\
= & \frac{1}{4}\left(e^{\theta+\theta'} + e^{\theta-\theta'} + e^{-\theta+\theta'} + e^{-\theta-\theta'}\right) && (9)\\
& + \frac{1}{4}\left(e^{\theta+\theta'} - e^{\theta-\theta'} - e^{-\theta+\theta'} + e^{-\theta-\theta'}\right) && (10)\\
= & \frac{1}{2}\left(e^{\theta+\theta'} + e^{-\theta-\theta'}\right) && (11)\\
= & cos(\theta+\theta') && (12)
\end{aligned}$$

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ii) Using the above formula show that two successive Lorentz transformations both along the x direction are such that if the velocity of transformation from the first to the second frame is v_1 and of the second to the third frame is v_2 then the velocity from the first to third frame is

$$\frac{v_f}{c} = \frac{\frac{v_1}{c} + \frac{v_2}{c}}{1 + \frac{v_1 v_2}{c^2}} \quad (13)$$

(Use the fact that $\tanh(\theta) = \frac{v}{c}$). Ie, while the rapidities (the "angle" in the hyperbolic function representation of the Lorentz transformations) of successive Lorentz transformations add, the velocities do not.

If

$$v_1 = \tanh(\theta) \tag{14}$$
$$v_2 = \tanh(theta') \tag{15}$$
$$v_f = \tanh(\theta+\theta') \tag{16}$$

$$\begin{aligned}
v_f &= \frac{sinh(\theta+\theta')}{\cosh(\theta+\theta')} && (17)\\
&= \frac{sinh(\theta)cosh(\theta') + cosh(\theta)sinh(\theta')}{cosh(\theta)cosh(\theta') + sinh(\theta)sinh(\theta')} && (18)\\
&= \frac{tanh(theta) + tanh(theta')}{1 + tanh(\theta)tanh(\theta')} && (19)\\
&= \frac{v_1 + v_2}{1 + v_1 v_2} && (20)
\end{aligned}$$

To show that the combined transformation is the transformation of th esum of the angles

$$\begin{aligned}
\tilde{x} &= cosh(\theta)x - sinh(\theta)t && (21)\\
\tilde{t} &= cosh(\theta)t - sinh(theta)x && (22)\\
\tilde{\tilde{x}} &= cosh(\theta')\tilde{x} - sinh(\theta')\tilde{t} && (23)\\
\tilde{\tilde{t}} &= cosh(\theta)\tilde{t} - sinh(theta)\tilde{x} && (24)\\
& && (25)
\end{aligned}$$

Substituting the first pair into the second, we get

$$\begin{aligned}
\tilde{\tilde{x}} &= cosh(\theta')\left(cosh(\theta)x - sinh(\theta)t\right) && (26)\\
&\quad + sinh(\theta')\left(cosh(\theta)t - sinh(theta)x\right) && (27)\\
&= \left(\cosh(\theta)\cosh(\theta') + \sinh(\theta)\sinh(\theta')\right)x && (28)\\
&\quad - \left(\sinh(\theta)\cosh(\theta') + \cosh(\theta)\sinh(\theta')\right)t && (29)\\
&= cosh(\theta+\theta')x - sinh(\theta+\theta')t && (30)
\end{aligned}$$

and similarly for $\tilde{\tilde{t}}$ Ie, combining two Lorentz transformations adds the rapidities.

2.) Show the consistancy of the special relativistic formulas. Consider the following synchronization of clocks. Alice and her friend Amy get together at the origin and sychronize their clocks to each other, ensuring that both show exactly the same time and run at the same rate. Now Amy very slowly (with a velocity δv approaching zero) moves away from Alice to a location X along Alice's x axis. Show that in Alice's frame, the time on Amy's clock at X will be synchronized with her clock. (Show that in the limit as δv goes to zero, the time difference between Amy's time to get to the location X and Alice's time for Amy to get to X are the same.) Now let us look at this process from Bob's point of view, who is moving with velocity v with respect to Alice. Show that Bob will calculate the difference between Amy's time to get to X and Alice's time is $\frac{vX}{c^2\sqrt{1-\frac{v^2}{c^2}}}$.

Hint, use the expression for the rate of Alice's and Amy's clocks according to Bob (time dilation) and look at the difference to lowest order in δv. Find the time it takes Amy to get to the point X and express the time difference in terms of X.

I will assume throughout that $c = 1$– ie that I have chosen coordinates for t so that t is measured in meters. To restore all the c's in the below, multiply att times by c, and divide all velocities by c.

In Alice's frame, Amy is travelling at velocity δv which is assumed to be very small. According to Alice, Amy's clock will tick at the slower rate $t_{Amy} = \sqrt{1-\delta v^2}t_{Alice}$. Now, to go a distance X, Amy must travel for a time of $\frac{X}{\delta v}$ (for Alice) which corresponds to a time of $\frac{\sqrt{1-\delta v^2}}{\delta v}$ for Amy. Expanding out in a series in δv we have that the time on Amy's clock when she gets there is $\frac{X}{\delta v} - \frac{1}{2}X\delta v$, while Alice will say she gets there in a time $\frac{X}{\delta v}$. Thus the difference between Amy's clock and Alice's clock is $-\frac{1}{2}X\delta v$ which goes to zero as δv gets very small. Ie, in the limit of very slow transport, Amy's time at the location X is the same as Alice's time there. Ie, This is a valid way of defining synchronization of clocks for Alice.

However, for Bob, Alice is moving at velocity v, and Amy will be moving at velocity $v+\delta v$ (assuming Amy moves in the same direction as Alice does with respect to Bob). Thus, Bob will say that Amy's clock goes at $\sqrt{1-(v+\delta v)^2}t_{Bob}$ while Alice's clock goes at $\sqrt{1-(v)^2}t_{Bob}$. Amy travels for a time of $\frac{X}{\delta v}$ in Amy's time, and thus in time $\frac{X}{\delta v\sqrt{1-(v+\delta v)^2}}$ in Bob's time. Thus after a time of $\frac{X}{\delta v}$ on Alice and Amy's clocks, Bob's time will be $\frac{X}{\delta v\sqrt{1-(v)^2}}$ for Alice's location and will be

$$\frac{X}{\delta v\sqrt{1-(v+\delta v)^2}} \tag{31}$$

$$\approx (\frac{X}{\delta v} + vX)\frac{1}{\sqrt{1-v^2}} \tag{32}$$

for Amy's clock. But this is just what we would expect since Bob's synchronization is different from Alice's. Ie, he will say that Amy and Alice's clocks are not synchronized by this procedure because of the time dilation of their two clocks.

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3. Let us assume that we measure time in units (light meters) such that the velocity of light is 1. Consider the following pairs of points. Are they timelike, spacelike or null separated and what is the squared distance between them?

i) $t=0;\ x=1;\ y=1;\ z=0$ and $t=1;\ x=1;\ y=0;\ z=0$

ii) $t=2;\ x=3;\ y=1\ ;z=0$ and $t=3;\ x=2;\ y=3;\ z=0$

iii) $t=3;\ x=1;\ y=1;\ z=0;$ and $t=5;\ x=1;\ y=0;\ z=0$

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i)$\Delta x^2 + \Delta y^2 + \delta z^2 - \Delta t^2 = 0^2 + (-1)^2 + 0^2 - 1^1 = 0$

These two points are null or light like separated.

ii)$\Delta x^2 + \Delta y^2 + \delta z^2 - \Delta t^2 = 4 > 0$

Sspacelike separated.

iii) $\Delta x^2 + \Delta y^2 + \delta z^2 - \Delta t^2 = -4 + 1 = -3 < 0$

Timelike separated.

In each case, what velocity and in what direction, would be needed to make the time separated points occur at the same point in space, or the spacelike separated points occur at the same time?

i)In the first case, it is impossible. These two points are light like separated. Ie, a light ray could travel between the two. Since the velocity of light is c in all frames, there is no way that these two points could either occur at the same point in space or at the same time in some frame.

ii) These are spacelike separated. We want a Lorentz transformation which will transform the Δt to zero. There are an infinite number of ways of doing this. For example, if we have a Lorentz boost in the y direction, we have

$$\delta\tilde{t} = \frac{1}{\sqrt{1-v^2}}(\Delta t - v\Delta y) \tag{33}$$

to make $\tilde{t}$ equal to zero, we need $v = \frac{\Delta t}{\Delta y} = \frac{1}{4}$. In general if we take the velocity in the $\vec{n}$ direction, we have

$$\delta\tilde{t} = \frac{1}{\sqrt{1-v^2}}(\Delta t - v\vec{n}\cdot\Delta\vec{x})$$

and we would need

$$v = \frac{\Delta t}{\vec{n}\cdot\Delta\vec{x}} \tag{34}$$

As long as $v < 1$ this would be a valid Lorentz transformation. Thus any $\vec{n}$ which obeys this would be a valid direction in which to take the Lorentz transformation and make the two events occur at the same time.

iii) These two points are timelike separated. Thus there exists a transformation so that the two events occur at the same place but different times. But since they are timelike separated, we could have a particle, travelling at velocity

$$\vec{v} = \frac{\Delta\vec{x}}{\Delta t} = (0, \frac{-1}{2}, 0) \tag{35}$$

travelling from one of those events to another. Thus if we went into the frame of that particle travelling at that velocity, the two events would occur at the same place, but different times.

4) i) Two particles are travelling at 4/5 the velocity of light and are 5 cm apart. How far apart are they in the frame in which they are at rest?

Boy, did I mess up this question. clearly if the two particles are separated in a direction perpendicular to their motion, they separation is independent of which frame (in the direction of their motion) I measure it in. Thus in this case in the rest frame they are 5 cm apart.

If on the other hand they are separated in the directionof their motion (ie one is travelling behind the other) then by the length contraction, their length in their rest frame would be

$$L' = \frac{5cm}{\sqrt{1-v^2}} = 25cm/3 = 8.33cm$$

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ii) Muons are created in the upper atmosphere of the earth (say 10km high). How fast would they have to be travelling so that half of them reached the surface of the earth?

Muons decay, and have a half life– ie a time over which half of the muon's will have decayed. I go to google to find out what that half life is, and come up with a whole slew of answers. 1.4μs, 1.52μsec, 2.2μsec, 3.7μsec.

This is an important lesson about Google. **Do not trust the answers you get from it**. Wikipedia gives 2.2μsec. for the lifetime. The Particle Physics Tables, which is the authoritative compilation put out by Lawrence Berkeley Labs every year give a mean life of $2.19703 \pm .00004$ μsec.

http://pdg.lbl.gov/2006/tables/contents_tables.html.

But the mean life, or lifetime is NOT the half life. The half life is related to the mean life by

half life=ln(2) mean life.= .693 mean life

The mean life is the time for 1/e of the muons to decay.

So the half life of the muon is 1.52μsec.

Now, if the muon is travelling at speed v, the earth time it takes to go through the atmosphere is $10km/v$ However for the muon's the time is only $\sqrt{1-(\frac{v}{c})^2}10km/v$. The muons decay to half their number in 2.2μ sec of their time. Thus

$$\sqrt{1-(\frac{v}{c})^2}10km/v = 1.5\ 10^{-6}sec \tag{36}$$

$$1-(\frac{v}{c})^2 = 2.210^{-20}v^2 \tag{37}$$

$$\frac{v^2 = 1}{1.1\ 10^{-17} + 2.210^{-20} \approx c^2} \tag{38}$$

Ie, we can write the above as

$$1-\frac{v}{c} \approx \frac{2.2\ 10^{-20}c^2}{2} = 10^{-3}$$

Ie, v is within .1% of c.

Note that from the point of view of the muons, it is not that their half life changes. They see that 10km of atmosphere as contracted by the Lorentz length contraction. So again they see that half their population has decayed while the thin slab of the atmosphere races by them at almost the speed of light.

5) Peter Spacerider has heard about Relativity and heard that from the point of view of a rapidly travelling observer, his own spaceship is really short. He passes a spaceship identical to his own travelling in the opposite directioni at almost the velocity of light. Just as the nose of his spaceship is at the tail of the other spaceship he presses the button in the nose to fire the laser canon in the tail of his own spaceship at the other spaceship. His collegue Johnny says "You are an idiot. It is the other spaceship that is really short since it is travelling with respect to us. Your shot has missed." Who is right? Why? What is wrong with the other's argument. (Note, you can assume that the distance between the spaceships passing each other is much less than the length of the spaceships.)

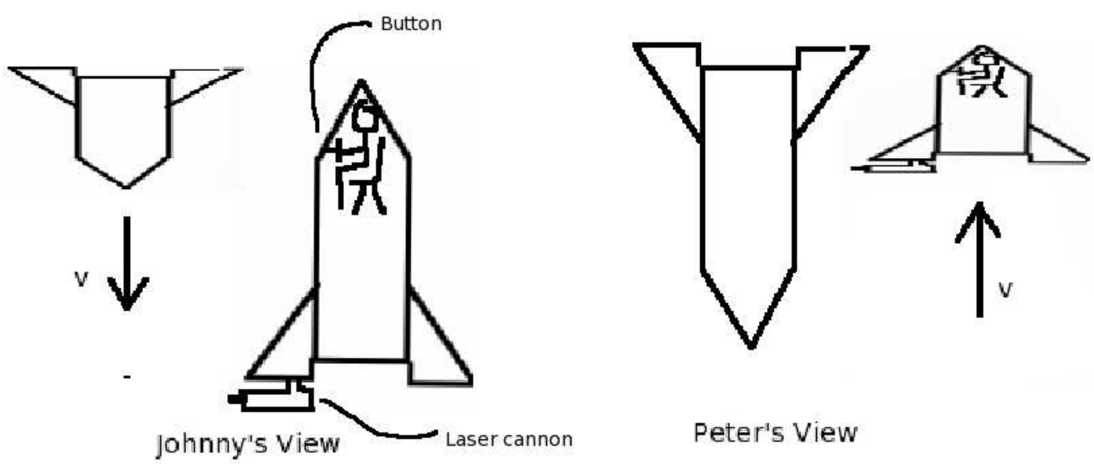

Figure 1: For Problem 5. The two views of the spaceships passing each other. Peter is in the nose of the right spaceship where he presses the button to fire the laser canon in the tail of his spaceship.

Johnny's argument is that Peter will miss the other spaceship because his shot will go in front of the other spaceship. This is always wrong. The key issue that both have forgotten is that it takes a finite time for the signal to pass down the ship to the canon. If we assume that the signal travels at the velocity of light, then Peter's viewpoint diagram shows that the signal will get to the canon at about the time that the canon has travelled almost to the tail of the other ship. Thus the shot will hit very near the tail of the other ship. In the case of the Johnny's point of view, since the other spaceship is travelling at only slightly slower than the signal (from the diagram, the contraction is just under 1/2 so the velocity of the other ship is about 90% of that of light). If the other ship were travelling at the velocity of light, then the shot would just hit exactly at the end of the other ship, since the signal and the tail of the spaceship would keep pace with each other.

This also tells us that if the signal travels slower than the other ship, then the the shot will miss– behind the other ship. In Peter's viewpoint, this will be because the signal is actually being carried along with the spaceship and thus reaches the back after the back of Peter's ship with its canon has passed the back of the other ship.