Physics 200

Taylor series

One of the key difference between physics and mathematics is the willingness of physics to tolerate approximations. There is a story from the early days of the Santa Fe institute in the 80s. The organisers decided to bring together physicists and economists to see if the thinking of the physicists could contribute to the very difficult problems that the economists were attacking. Both groups were shocked. The physicists were shocked at the rigour of the economists. The theoretical economists would set up assumptions, prove lemmas, theorems, and corrolaries. The economists were shocked at the sloppiness of the physicists– the physicists seemed willing to make what seemed to be gross approximations, to slap together crude computer programs to see what the consequences of some ideas would be.

One of the key tools of approximation schemes is the taylor series. We know from mathematics that given any function $f(x)$ we can expand the function near some initial point x_0 by the series

$$f(x) = f(x_0) + \frac{1}{1!}\frac{df(x_0)}{dx}(x - x_0) + ... + \frac{1}{n!}\frac{d^n f(x_0)}{dx^n}(x - x_0)^n + ... \tag{1}$$

More importantly, it is often true (but not always) that if we truncate the series at some value of n the difference between the true function f(x) and the approximation formed by the first n terms in the series goes as order $(x - x_0)^{n+1}$. Ie, that in the limit as $x \to x_0$, then

$$\frac{1}{(x - x_0)^{n+1}}\left(f(x) - \sum_{m=0}^{n}\frac{1}{m!}\frac{d^m f(x_0)}{dx^m}(x - x_0)^m\right) \to \text{Constant} \tag{2}$$

In fact what physicists would usually do is to keep only the lowest order terms– ie, to choose n so that the final answer contains only one term in the taylor expansion. For example, if $f(x_0) = 0$, but the first derivative is not, then the lowest order terms would be $n = 1$.

$$\frac{1}{\sqrt{1 - v^2} \approx 1 + \frac{1}{2}v^2 + O(v^4).} \tag{3}$$

sometimes this would be approximated by the first two terms, while in other problems one would choose 1 to approximate this function. It would depend on the final answer. For example, if choosing the value 1 as the approximation led to an answer of zero, one would try using the next term as well to see if it resulted in a non-zero answer.

Note one should always be consistant. If in some terms one keeps terms only to linear order in v, one should not keep terms to second order in v in some other term (unless there were a strong reason to do so. For example

$$v + 2\frac{\sqrt{1-v^2}-1}{v} \approx \frac{1}{4}v^3 \tag{4}$$

One must, because of the division by v in the second term, keep terms in the expansion of $\sqrt{1-v^2}$ to order v^4. However such cases are usually clear.

We note that if we have a function of v^2, then the taylor series of $g(v^2)$ in terms of v about $v = 0$ is exactly same as the taylor series in terms of the variable v^2. Thus

$$g(v^2) = g(0) + g'(0)v^2 + ... \tag{5}$$

ie,

$$g(f(x)) = \sum_{m=0} \frac{1}{m!} g^{(n)}(f(x_0)(\sum_{r=1} \frac{1}{r!} f^{(r)}(x_0)(x-x_0)^r)^m \tag{6}$$

where $g^{(n)}$ is the n^{th} derivative of g with respect to its argument.