Physics 200-05
Practice 3

1. Given the two matricees

$$A = \begin{pmatrix} 0 & 7 & 5 \\ -2 & 1 & 2 \\ 1 & 1 & 0 \\ -2 & -1 & 1 \end{pmatrix} \tag{1}$$

$$B = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \tag{2}$$

Do the following matrices exist? If they do show what they are. AB, BA, A^T,B^T,A^TB, $A+B$, $B+B$, $2B$.

BA and A^TB do not exist, because the first matrix has a different number of colums than the second has rows.

$A+B$ does not exist because both must have the same number of rows and columns.

$$AB = \begin{pmatrix} 14 & 12 & 19 \\ 2 & 1 & 4 \\ 2 & 2 & 2 \\ -2 & -2 & -1 \end{pmatrix} \tag{3}$$

$$A^T = \begin{pmatrix} 0 & -2 & 1 & -2 \\ 7 & 1 & 1 & -1 \\ 5 & 2 & 0 & 1 \end{pmatrix} \tag{4}$$

$$B^T = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \tag{5}$$

$$B + B = 2B = \begin{pmatrix} 0 & 4 & 0 \\ 2 & 2 & 2 \\ 0 & 4 & 2 \end{pmatrix} \tag{6}$$

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2. Show that

$$A = \begin{pmatrix} cosh(\mu) & sinh(\mu) & 0 & 0 \\ sinh(\mu) & cosh(\mu) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & \\ 0 & cos(\theta) & sin(\theta) & 0 & (8) \\ 0 & -sin(\theta) & cos(\theta) & 0 & \\ 0 & 0 & 0 & 1 & \end{pmatrix} \tag{9}$$

are both Lorentz transformations (with the time having been chosen so that $c =$ 1.). Show that $(GB^TG)AB$ is also a Lorentz transformation, and corresponds to an ordinary transformation along the direction at θ degrees to the x axis.

If L is a Lorentz transformation then $L^TGL = G$ where

$$G = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10}$$

Then

$$A^TGA = \begin{pmatrix} -cosh(\mu) & -sinh(\mu) & 0 & 0 \\ sinh(\mu) & cosh(\mu) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} A \tag{11}$$

$$= \begin{pmatrix} -cosh(\mu)^2 + sinh(\mu)^2 & 0 & 0 & 0 \\ 0 & cosh(\mu)^2 - sinh(\mu)^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = G \tag{12}$$

and

$$B^TGB = \begin{pmatrix} -1 & 0 & 0 & 0 & \\ 0 & cos(\theta) & -sin(\theta) & 0 & (13) \\ 0 & sin(\theta) & cos(\theta) & 0 & \\ 0 & 0 & 0 & 1 & \end{pmatrix} B \tag{14}$$

$$= \begin{pmatrix} -1 & 0 & 0 & 0 & \\ 0 & cos(\theta)^2 + sin(\theta)^2 & 0 & 0 & \\ & 0 & 0 & cos(\theta)^2 + sin(\theta)^2 & 0 \\ 0 & 0 & 0 & 1 & \end{pmatrix} = G \tag{15}$$

Using $G^T = G$ and $GG = I$ we have

$$((GB^TG)AB)^TG(GB^TG)AB = B^TA^TG^TBG^TG(GB^TG)AB = B^TA^TGBGB^TGAB \tag{16}$$

Now, if B^T is also a Lorentz transformation (it is just B with $\theta \rightarrow theta$ so the above becomes

$$B^TA^TG(BGB^T)GAB = B^TA^TGGGAB = B^TA^TGAB = B^TGB = G \tag{17}$$

Ie, it is a Lorentz transformation.

$$GB^TGAB = B^TAB = \begin{pmatrix} cosh(\mu) & sinh(\mu) & 0 & 0 \\ cos(\theta)cosh(\mu) & cos(\theta)sinh(\mu) & -sin(\theta) & 0 \\ sin(\theta)cosh(\mu) & sin(\theta)sinh(\mu) & cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} B \quad (18)$$

$$= \begin{pmatrix} cosh(\mu) & sinh(\mu)cos(\theta) & sinh(\mu)sin(\theta) & 0 & (19) \\ sinh(\mu)cos(\theta) & cosh(\mu)cos(\theta)^2 + sin(\theta)^2 & cos(\theta)sin(\theta)(cosh(\mu)-1) & 0 & (20) \\ sinh(\theta)sun(\theta) & cos(\theta)sin(\theta)(cosh(\mu)-1) & cosh(\mu)sin(\theta)^2 + cos(\theta)^2 & 0 & (21) \\ 0 & 0 & 0 & 1 & \end{pmatrix} \quad (22)$$

But, the Lorentz transformation in the direction θ is

$$\tilde{t} = cosh(\mu)t + sinh(\mu)(cos(\theta)x + sin(\theta)y) \quad (23)$$

$$(cos(\theta)\tilde{x} + sin(\theta)\tilde{y}) = cosh(\mu)(cos(\theta)x + sin(\theta)y) + sinh(\mu)t \quad (24)$$

$$cos(\theta)\tilde{y} - sin(\theta)\tilde{x} = cos(\theta)y - sin(\theta)x \quad (25)$$

$$\tilde{z} = z \quad (26)$$

If we solve this for $\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}$ we find that this is just the above transformation.

3.) Bob heads off for the nearest star (Alpha Centauri) which is 4 light years away at a velocity of .9 c. He arrives and then discovers that he left his key to the house on earth, and immediately returns at .95c. How long will he have been gone from earth according to the people on earth and how long with respect to Bob himself.

According to Alice, his proper time per unit time in Alice's frame is $\sqrt{\Delta t^2 - \Delta x^2}$. But $\Delta x = .9\delta t$. Thus the proper time along the path on the outward trip is $\sqrt{1 - .9^2}\Delta t$. Since the distance is 4 light years and the speed is .9, the time is 4/.9 years= 4.444 years. The proper time along the outward path is thus

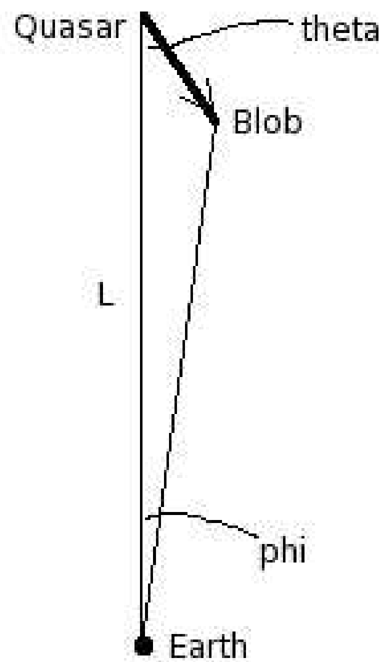
$$\tau_1 = \frac{4}{.9}\sqrt{1 - .81} = 4(.484) = 1.94\text{years} \quad (27)$$

On the return path the time will be

$$\tau_2 = \frac{4}{.95}\sqrt{1 - .95^2} = 4(.329) = 1.31 \text{ years} \quad (28)$$

for a total time of 3.25 years.

4.) How much shorter or longer is the track according to the runner for the 100m dash if the runner runs at 10m/s?



The length contraction is given by

$$\tilde{L} = \sqrt{1 - \frac{v^2}{c^2}} L = \sqrt{1 - (3.3 \cdot 10^{-8})^2} 100 \approx 100m(1 - \frac{1}{2} 10^{-15}) \tag{29}$$

Thus the track will be approximately half of a femtometer shorter for the sprinter. (that is about the diameter of a nucleus).

5.) A quasar ejects a blob of material at .9c Assume that the quasar is a distance L away from the earth and that the material is ejected at an angle θ with respect to the direction from the quasar to the earth. What is the rate of change of the angle ϕ of the blob as seen from the earth as a function of θ? What is the maximum value of this angular change in position as a function of θ? If you ascribe a transverse velocity to this material by $v_T = L\frac{d\phi}{dt}$, how large can v_T be. (assume that L is very large, much larger than the distance between the blob and the quasar.) (Note that you must take into account the propagation of light from the blob to the observer).

Let us look at how far the blob moves in one second. In the transverse direction to the earth, it will move $v\sin(\theta)$ meters. It will also move $v\cos(\theta)$ closer to the earth. Thus the distance the light has to travel is less by $L-v\cos(\theta)$ than before that one second. The light at the beginning of the one second takes

L/c to arrive at the earth, while that after the second takes $(L - v\cos(\theta))/c$ to arrive at the earth. The difference in times will thus be

$$\Delta t = 1 - \frac{v}{c} cos(\theta) \tag{30}$$

The change in angle over this time will be $\delta\phi = \frac{v\sin(\theta)}{L}$

The apparent transverse speed of the blob will be

$$\frac{L\delta\phi}{\Delta t = v\frac{\sin(\theta)}{1-\frac{v}{c}cos(\theta)}} \tag{31}$$

This is maximized at the angle θ given by

$$\frac{d}{d\theta}\frac{\sin(\theta)}{1 - \frac{v}{c}cos(\theta)} = 0 \tag{32}$$

$$cos(\theta)(1 - \frac{v}{c}cos(\theta)) = sin(\theta)^2 \tag{33}$$

$$cos(\theta) = \frac{1 - \sqrt{5 - 4\frac{v}{c}}}{1 - \frac{v}{c}} \tag{34}$$

$$\tag{35}$$

For $\frac{v}{c} = .9$ this gives $cos(\theta) = .916$ and the transverse velocity is $v_T = 2.28c$ Ie, the blob appears to be moving much faster than light.

When this was first observed it confused physicists no end.
