Physics 200-05
Practice 1

1). Show explicitly that if

$$\tilde{x} = cos(\theta)x + sin(\theta)y \tag{1}$$
$$\tilde{y} = cos(\theta)y - sin(\theta)x \tag{2}$$

then the distance from the origin in the $\tilde{x}$, $\tilde{y}$ system is the same as in the x, y system.

$$\tilde{dist}^2 = \tilde{x}x^2 + \tilde{y}^2 \tag{3}$$
$$= (x\cos(\theta) + y\sin(\theta))^2 + (y\cos(\theta) - x\sin(\theta))^2 \tag{4}$$
$$= x^2(\cos(\theta)^2 + \sin(\theta)^2) + y^2(\sin(\theta)^2 + \cos(\theta)^2) \tag{5}$$
$$+ 2xy(cos(\theta)sin(\theta) - sin(\theta)cos(\theta)) \tag{6}$$
$$= x^2 + y^2 = dist^2 \tag{7}$$

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2) [Hard] Show that if $\tilde{x} = X(x,y)$, $\tilde{y} = Y(x,y)$, then the requirement that the distance between any two nearby points x_1, y_1 and x_2, y_2 be the same as between $\tilde{x}_1, \tilde{y}_1$ and $\tilde{x}_2, \tilde{y}_2$, for all x_1, y_1 and nearby x_2, y_2 is that

$$\left(\frac{\partial X}{\partial x}\right)^2 + \left(\frac{\partial Y}{\partial x}\right)^2 = \left(\frac{\partial X}{\partial y}\right)^2 + \left(\frac{\partial Y}{\partial y}\right)^2 = 1 \tag{8}$$
$$\left(\frac{\partial X}{\partial x}\right)\left(\frac{\partial X}{\partial y}\right) = -\left(\frac{\partial Y}{\partial y}\right)\left(\frac{\partial Y}{\partial x}\right) \tag{9}$$

and that the second derivatives of X and Y are all zero. (Expand the expression for the distance in the $\tilde{x}, \tilde{y}$ coordinates in a taylor series in terms of the the $x_2 - x_1$ and $y_2 - y_1$).

This means that the only transformations must be of the form

$$X(x,y) = cos(\theta)x + sin(\theta)y + c_x \tag{10}$$
$$Y(x,y) = cos(\theta)y - sin(\theta)x + c_y \tag{11}$$

where the c_x and c_y are constants and

$$cos(\theta) = \frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y}. \tag{12}$$

Ie, in two dimensions, the only transformations of the coordinates which keeps all distances the same are rotations and translations.

$$\tilde{dist}^2 = (\tilde{x}_2 - \tilde{x}_1)^2 + (\tilde{y}_2 - \tilde{y}_1)^2 \tag{13}$$
$$((X(x_2,y_2) - X(x_1,y_1))^2 + (Y(x_2,y_2) - Y(x_1,y_1))^2 \tag{14}$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 \tag{15}$$

where the last equal sign comes from what we want. Ie, we want both distances to be the same for all values of x_1, y_1, x_2, y_2. Let me make typing easier and write x, y for x_1, y_1

Take the second derivative of both sides of this expression with respect to x_2, y_2 evaluated at $x_2 = x$ and $y_2 = y$. (The first derivative is zero) First taking the second derivative with respect to x_2 we get

$$2\left(\frac{\partial X}{\partial x}\right)^2 + \left(\frac{\partial Y}{\partial x}\right)^2 = 2 \tag{16}$$

Where X is shorthand for $X(x, y)$ and similarly for Y.

Taking the second derivative with respect to $y2$ and evaluating at $y2 = y$, we get

$$2\left(\frac{\partial X}{\partial y}\right)^2 + \left(\frac{\partial Y}{\partial y}\right)^2 = 2 \tag{17}$$

Finally taking the mixed derivative, first wrt to x_2 and then with respect to y_2 we get

$$2\left(\frac{\partial X}{\partial x}\frac{\partial X}{\partial y}\right) + 2\left(\frac{\partial Y}{\partial x}\frac{\partial Y}{\partial y}\right) = 0 \tag{18}$$

From the first equation, we know that $\frac{\partial X}{\partial x}$ and $\frac{\partial Y}{\partial x}$ are both less than 1, and their squares sum up to 1. We can thus take

$$\frac{\partial X}{\partial x} = cos(\theta) \tag{19}$$

$$\frac{\partial Y}{\partial x} = -sin(\theta) \tag{20}$$

for some value of θ. θ may well depend on x, y. Similarly

$$\frac{\partial X}{\partial y} = sin(\phi) \tag{21}$$

$$\frac{\partial Y}{\partial x} = cos(\phi) \tag{22}$$

For some value of ϕ from the second equation. Then the third equation becomes

$$\left(\frac{\partial X}{\partial x}\frac{\partial X}{\partial y}\right) + 2\left(\frac{\partial Y}{\partial x}\frac{\partial Y}{\partial y}\right) \tag{23}$$

$$= cos(\theta)\sin(\phi) - \sin(\theta)cos(\phi) = sin(\theta - \phi) = 0 \tag{24}$$

Thus $\phi = \theta$.

We now have to ask if θ can depend on x, y. We will use the two equations.

$$\frac{\partial}{\partial y}\frac{\partial X}{\partial x} = \frac{\partial}{\partial x}\frac{\partial X}{\partial y} \tag{25}$$

$$\frac{\partial}{\partial y}\frac{\partial Y}{\partial x} = \frac{\partial}{\partial x}\frac{\partial Y}{\partial y} \tag{26}$$

Ie, for both Y and X the order in which one takes the mixed derivatives does not matter.

But writing the first derivatives in terms of θ derived above, and using the chain rule for the derivatives of cos and sin, we get

$$\frac{\partial}{\partial y}cos(\theta) = \frac{\partial}{\partial x}sin(\theta) \tag{27}$$

$$\frac{\partial}{\partial y}(-sin(\theta) = \frac{\partial}{\partial x}cos(\theta) \tag{28}$$

or

$$-sin(\theta)\frac{\partial\theta}{\partial y} = cos(\theta)\frac{\partial\theta}{\partial x} \tag{29}$$

$$-cos(\theta)\frac{\partial\theta}{\partial y} = -sin(\theta)\frac{\partial\theta}{\partial x} \tag{30}$$

Multiplying the first by $\sin(\theta)$ and the second by $\cos(\theta)$ and adding we get

$$-(sin(\theta)^2 + cos(\theta)^2)\frac{\partial\theta}{\partial y} = 0 \tag{31}$$

which says that θ must be independent of y. Thus θ must also be independent of x and θ must be constant.

Thus all second and higher deriatives of X and Y must be zero.

X and Y must be linear functions of x and y and knowing what the first derivatives are, we get

$$X(x, y) = cos(\theta)x + sin(\theta)y + c_X \tag{32}$$

$$Y(x, y) = -sin(\theta)x + cos(\theta)y + c_Y \tag{33}$$

where c_X and c_Y are integration constants.

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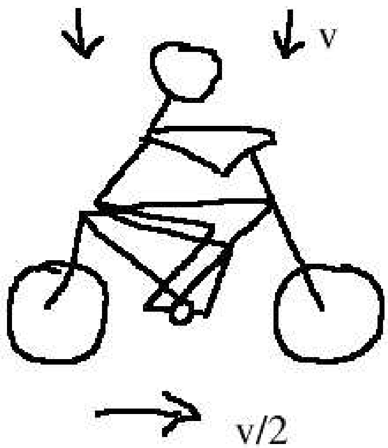
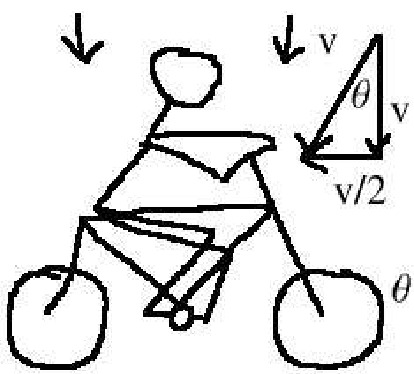
3) (Abberation) Rain is falling vertically and hits the ground with speed c. A bicyclist is travelling through the rain with velocity $c/2$. At what angle (from the vertical) does the cyclist feel the rain as hitting him?

In the frame of the bike, the rain comes down with velocity v and moves toward the bicyclist with velocity v/2. The angle θ is given by

$$tan(\theta) = \frac{v/2}{v} = \frac{1}{2} \tag{34}$$


v
v/2


v
θ
v
v/2
θ

from which $\theta = .463 = 26.6$ degrees.

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4)Assume that the aether is completely dragged by light. Thus the velocity of light in water flowing with the light is $\frac{c}{n} + v$ while that for light in water flowing against the light is $\frac{c}{n} - v$. What would be the difference in the time (to lowest order in v) it takes light to traverse two meters of flowing water, if the water is flowing at 10 m/sec. (recall that the velocity of light, $c = 3 \cdot 10^8 m/sec$ and the index of refraction of water is 1.3. If the frequency of light used is $2 \cdot 10^{15} Hz$ what is this difference in time as a fraction of the period of the light.

Fresnels theory says that the drag is not v, but rather is (to lowest order in v) $v(1 - \frac{1}{n^2})$. How much of a difference would this make in the above exeriment?(See text book).

The time on the trip with the flowing water would be $\frac{L}{c/n-v}$ where L is the length of the path. The time against the flowing water would be $\frac{L}{c/n+v}$. Thus the difference in time would be

$$\frac{L}{c/n - v} - \frac{L}{c/n + v} = \frac{2Lv}{(c/n)^2 - v^2} \tag{35}$$

keeping only to lowest order in v (ie, expanding in a taylor series in v and keeping only the first term) we get that the difference in time is $\frac{2Ln^2v}{c^2}$. Plugging in the values for L, n, c and v we get .75 10^{-15} sec. Since the number of cyles per second of the light is 2 10^{15} Hz, this corresponds to 1.5 cycles.

In the case of the partial drag, the drag velocity instead of v is $\frac{v(1-1}{n^2)=.41v}$ Thus the time difference is

$$\frac{2Lv(1 - \frac{1}{n^2})}{(c/n)^2 - v^2} \tag{36}$$

which gives a number of cycles of .62 for the time difference.