Physics 200-04
Clarifications on Lecture 1

Gravitational force

Looking at the equation for the motion of a body a under the Gravitational force from body b.

$$M^a \frac{d^2\vec{x}^a}{dt^2} = \vec{F} = -GM^aM^b \frac{(\vec{x}^a - \vec{x}^b)}{|\vec{x}^a - \vec{x}^b|^3} \tag{1}$$

where $|\vec{x}^a - \vec{x}^b|$ is the distance between body a and body b. Some were a bit confused as to this way of writing the force, since they recalled that the force went as $\frac{1}{r^2}$. This expression gives both that magnitude of the force **and** the direction of the force. The vector,

$$\frac{(\vec{x}^a - \vec{x}^b)}{|\vec{x}^a - \vec{x}^b|} \tag{2}$$

is a vector with a unit length (the denominator is just the length of the vector in the numerator) which points in the direction of the line joining particle b to particle a. Ie this expression captures both the usual magnitude of the gravitational force, and its direction.

Note that in my blackboard notes I think I missed the minus sign in the Force above. The force on a is of course directed toward particle b.

Note also that there is also the equation for the second particle b as well. When one does the trasformation (rotation), one has to rotate both a and b's locations at the same time (ie, both $\vec{x}^a$ and $\vec{x}^b$ have to be rotated at the same time for the force to be rotationally invariant.)

Change of coordinates

Some were a bit confused with what the $\tilde{x}, \tilde{y}$ were. Under a change (eg a rotation) the coordinates, the numbers used to describe the location of the particle, will change. (This could either be because of an active change– ie the bodies were in some sense moved physically by the roatation, or by what is called a passive, where one simply changes what one calls the x axis and the y axis. The x, y coordinates are the old numbers which descibe where the particles were, while the $\tilde{x}, \tilde{y}$ are the new numbers which describe where the particles are (eg with respect to the new axes). In both cases we assume that the distance of the particles from the origin obey the same relation $x^2 + y^2$ and $\tilde{x}^2 + \tilde{y}^2$. Now, the location of the particles in the old coordinate system will be some function of what the coordinates are in the new system. Ie, x and y will be some sort of function of $\tilde{x}$ and $\tilde{y}$. I assumed the simplest such function, that

$$x = \alpha\tilde{x} + \beta\tilde{y} \tag{3}$$
$$y = \gamma\tilde{x} + \delta\tilde{y} \tag{4}$$

One could imagine trying a more complicated equation relating the values of the coordinates of the positions in the two systems of coordinates. However, it is should be hard to see how the distance could remain the same form for all values of x, y. Eg, if there were some quadratic term in the relation, then plugging in these relations into $x^2 + y^2$ would lead to quartic (eg $\tilde{x}^4$) type terms. It can in fact be proven that such a linear relation is the most general relation possible which preserves the length function for all points. In the lecture we then showed that in order to preserve the length function, the only possible linear relation was

$$x = \cos(\theta)\tilde{x} - \sin(\theta)\tilde{y} \tag{5}$$

$$y = \sin(\theta)\tilde{x} + \cos(\theta)\tilde{y} \tag{6}$$

This is a rotation. In fact the definition of a rotation is a transformation of the coordinates of a particle which leaves the distance invariant, no matter where the particle is located.

Thus, Newton's second law is invariant under rotations IF the direction of the force changes under a rotation in the same was the position vector does, and if the force depends on quantities– like masses, constants like G and distances– which do not change under rotation.

sinh and cosh

Just in case I do not get to where I want to be at the end of Friday's lecture, the definitions of the functions sinh and cosh (the hyperbolic sin and cos) are given by the equations

$$\sinh(\theta) = \frac{e^\theta - e^{-\theta}}{2} \tag{7}$$

$$\cosh(\theta) = \frac{e^\theta + e^-\theta}{2} \tag{8}$$

These functions have many of the properties of the sin and cos functions. For example the functions obey

$$\frac{d^2 \sinh(\theta)}{d\theta^2} = \sinh(\theta) \tag{9}$$

while the ordinary sin obeys

$$\frac{d^2 sin(\theta)}{d\theta^2} = -\sin(\theta) \tag{10}$$

(Note the change in sign on the right hand side). It is also easy to show that

$$\cosh(\theta)^2 - \sinh(\theta)^2 = 1 \tag{11}$$

contrasting with the trigonometric relation

$$\cos(\theta)^2 + \sin(\theta)^2 = 1 \tag{12}$$

Note that in the case of the hyperbolic functions, they are not periodic in their arguments. In fact as θ gets large, both $\cosh(\theta)$ and $\sinh(\theta)$ get very large.