Physics 200-05

Assignment 8

1. Consider the two two-particle states given by

$$|\psi\rangle = N_\psi\left(|+1;1\rangle \otimes |-1;1\rangle + |-1;1\rangle \otimes |+1;1\rangle\right) \quad (1)$$
$$|\phi\rangle = N_\phi\left(|+1;1\rangle \otimes |+1;1\rangle + |-1;1\rangle \otimes |-1;1\rangle\right) \quad (2)$$

where $|+1;1\rangle$ means the eigenstate with the $+1$ eigenvalue for the attribute Σ_1 in the case of the first particle and Ξ_1 for the second. What are possible values for the normalisation factors N_ψ and N_ϕ? What is the inner product $\langle\phi||\psi\rangle$.

(Do not try to expand out the direct products in terms of matrices.)

[2 – NOte that if, despite my stricture, they do expand it out as a component direct product, subtract just 1 mark on yhr question]

$$\begin{aligned}
\langle\psi||\psi\rangle &= N_\psi^* N_\psi\left(\langle +1;1| \otimes \langle -1;1| + \langle -1;1| \otimes \langle +1;1|\right) \\
&\qquad \left(|+1;1\rangle \otimes |-1;1\rangle + |-1;1\rangle \otimes |+1;1\rangle\right) \\
&= |N_\psi|^2\left(\langle +1;1| \otimes \langle -1;1||+1;1\rangle \otimes |-1;1\rangle + \langle +1;1| \otimes \langle -1;1||-1;1\rangle \otimes |+1;1\rangle\right. \\
&\qquad \left. + \langle -1;1| \otimes \langle +1;1||+1;1\rangle \otimes |-1;1\rangle + \langle -1;1| \otimes \langle +1;1||-1;1\rangle \otimes |+1;1\rangle\right) \\
&= |N_\psi|^2\left(\langle +1;1||+1;1\rangle\langle -1;1||-1;1\rangle + \langle +1;1||-1;1\rangle\langle -1;1||+1;1\rangle\right. \\
&\qquad \left. + \langle -1;1||+1;1\rangle\langle +1;1||-1;1\rangle + \langle -1;1||-1;1\rangle\langle +1;1||+1;1\rangle\right) \\
&= |N_\psi|^2\left(1\cdot 1 + 0\cdot 0 + 0\cdot 0 + 1\cdot 1\right) = 2N^*N \qquad (3)
\end{aligned}$$

Thus in order that the vector have unit norm, we need $N^*N = 1/2$ of can choose $N_\psi = \frac{1}{\sqrt{2}}$.

Carrying out exactly the same for $|\phi\rangle$ we have

[1– Of course if the main technique is shown for this case instead of the first, give this 2 and the first 1]

$$\begin{aligned}
\langle\phi||\phi\rangle &= N_\phi^* N_\phi\left(\langle +1;1| \otimes \langle 1;1| + \langle -1;1| \otimes \langle -1;1|\right) \\
&\qquad \left(|+1;1\rangle \otimes |+1;1\rangle + |-1;1\rangle \otimes |-1;1\rangle\right) \\
&= |N_\phi|^2\left(\langle +1;1| \otimes \langle +1;1||+1;1\rangle \otimes |+1;1\rangle + \langle +1;1| \otimes \langle +1;1||-1;1\rangle \otimes |-1;1\rangle\right. \\
&\qquad \left. + \langle -1;1| \otimes \langle -1;1||+1;1\rangle \otimes |+1;1\rangle + \langle -1;1| \otimes \langle -1;1||-1;1\rangle \otimes |-1;1\rangle\right) \\
&= |N_\phi|^2\left(\langle +1;1||+1;1\rangle\langle +1;1||+1;1\rangle + \langle +1;1||-1;1\rangle\langle +1;1||-1;1\rangle\right. \\
&\qquad \left. + \langle -1;1||+1;1\rangle\langle -1;1||+1;1\rangle + \langle -1;1||-1;1\rangle\langle -1;1||-1;1\rangle\right) \\
&= |N_\phi|^2\left(1\cdot 1 + 0\cdot 0 + 0\cdot 0 + 1\cdot 1\right) = 2N_\phi^* N_\phi \qquad (4)
\end{aligned}$$

From which we take $N_\phi = \frac{1}{\sqrt{2}}$

[2]

$$\begin{aligned}
\langle\phi||\psi\rangle &= N_\phi^*\left(\langle +1;1| \otimes \langle +1;1| + \langle -1;1| \otimes \langle -1;1|\right) \\
&\qquad N_\psi\left(|+1;1\rangle \otimes |-1;1\rangle + |-1;1\rangle \otimes |+1;1\rangle\right) \\
&= \frac{1}{2}\left(\langle +1;1| \otimes \langle +1;1||+1;1\rangle \otimes |-1;1\rangle + \langle +1;1| \otimes \langle +1;1||-1;1\rangle \otimes |+1;1\rangle\right. \\
&\qquad \left. + \langle -1;1| \otimes \langle -1;1||+1;1\rangle \otimes |-1;1\rangle + \langle -1;1| \otimes \langle -1;1||-1;1\rangle \otimes |+1;1\rangle\right) \\
&= \frac{1}{2}\left(\langle +1;1||+1;1\rangle\langle +1;1||-1;1\rangle + \langle +1;1||-1;1\rangle\langle +1;1||+1;1\rangle\right. \\
&\qquad \left. + \langle -1;1||+1;1\rangle\langle -1;1||-1;1\rangle + \langle -1;1||-1;1\rangle\langle -1;1||+1;1\rangle\right) \\
&= \frac{1}{2}(1\cdot 0 + 0\cdot 1 + 0\cdot 1 + 1\cdot 0) = 0
\end{aligned} \tag{5}$$

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2. If Σ_1 is the Pauli spin matrix for the first particle with matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and Ξ_1 is the Pauli spin matrix for the second particle with the same matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, show in problem 1 that both $|\psi\rangle$ and $|\phi\rangle$ are eigenvectors of the matrix $\Sigma_1 \otimes \Xi_1$. What are the eigenvalues?

$$\Sigma_1|1;1\rangle = +1|1;1\rangle \tag{6}$$
$$\Sigma_1|-1;1\rangle = -1|-1;1\rangle \tag{7}$$

And similarly for Ξ_1.

[1 – understanding that these are the eigenvectors of Σ_1 and Ξ_1]

Then [2– including understanding how the direct product distributes itself amongst the states]

$$\begin{aligned}
\Sigma_1 \otimes \Xi_1|\psi\rangle &= \frac{1}{\sqrt{2}}\left(\Sigma_1 \otimes \Xi_1|+1;1\rangle \otimes |-1;1\rangle + \Sigma_1 \otimes \Xi_1|-1;1\rangle \otimes |+1;1\rangle\right) \\
&\qquad \text{because of the distributed law of matrix multiplication} \\
&= \frac{1}{\sqrt{2}}\left((\Sigma_1|+1;1\rangle) \otimes (\Xi_1|-1;1\rangle) + (\Sigma_1|-1;1\rangle) \otimes (\Xi_1|1;1\rangle)\right) \\
&\qquad \text{by the definition of matrix multiplication with the direct product}
\end{aligned}$$

$$= \frac{1}{\sqrt{2}}(((+1)|+1;1\rangle) \otimes (-1)|-1;1\rangle) + ((-1)|-1;1\rangle) \otimes ((+1)|1;1\rangle))$$

by the eigenvector/value rule

$$= (-1)\frac{1}{\sqrt{2}}(|+1;1\rangle \otimes |-1;1\rangle) + |-1;1\rangle \otimes |1;1\rangle) = -|\psi\rangle \tag{8}$$

[1] Thus $|\psi\rangle$ is an eigenvectory of $\Sigma_1 \otimes \Xi_1$ with eigenvalue -1.

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3. Show expicitly that if

$$|1;3\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{9}$$

$$|-1;3\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{10}$$

for each of the two particles, then the four dimensional vector

$$|1;3\rangle \otimes |1;3\rangle + |-1;3\rangle \otimes |-1;3\rangle \tag{11}$$

cannot be writen as a simple product

$$|\chi\rangle \otimes |\xi\rangle \tag{12}$$

for any choice of $|\chi\rangle$ and $|\xi\rangle$. Such a vector for two particles which cannot be written as a simple product of vectors for the two single particles is called an entangled state.

(In this case do expand the direct product in terms of matrices.)

[**Note:** in the physics literature, that $\otimes$ symbol is almost always omitted. Thus $|\chi\rangle \otimes |\xi\rangle$ is written as $|\chi\rangle|\xi\rangle$ and you are expected to know that it is the direct product that is being used if you are refering to two separate particles. Similarly $\Sigma_1 \otimes \Xi_1$ is written as $\Sigma_1\Xi_1$ where you are to remember that this is a direct product not a matrix product because Σ_1 and Ξ_1 belong to two separate particles. It is never correct to matrix multiply attributes which belong to separate particles.

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[1]

$$|1;3\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{13}$$

$$|-1;3\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{14}$$

Thus

[1 – note that the order – ie whether the first or second is first is not important, as long as it is consistant.]

$$
\begin{aligned}
|1;3\rangle \otimes |1;3\rangle \quad + \quad |-1;3\rangle \otimes |-1;3\rangle &= \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} && (15)\\
&= \begin{pmatrix} 1\begin{pmatrix}1\\0\end{pmatrix} \\ 0\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} + \begin{pmatrix} 0\begin{pmatrix}0\\1\end{pmatrix} \\ 1\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix} && (16)\\
&= \begin{pmatrix}1\\0\\0\\1\end{pmatrix} && (17)
\end{aligned}
$$

[1]Now however,

$$
\begin{aligned}
\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix} \otimes \begin{pmatrix}\chi_1\\\chi_2\end{pmatrix} &= \begin{pmatrix} \chi_1\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix} \\ \chi_2\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}\end{pmatrix} && (18)\\
&= \begin{pmatrix}\chi_1\xi_1\\\chi_1\xi_2\\\chi_2\xi_1\\\chi_2\xi_2\end{pmatrix} && (19)
\end{aligned}
$$

[2– arguing through the inconsistancy]

[Note that there may be other ways of doing this.]Trying to equate this with the previous term, we have that $\chi_1\xi_2 = 0$.which means that either $\chi_1 = 0$ or $\xi_2 = 0$. But if $\chi_1 = 0$ then the first component must be 0 as well, and it is 1, while if $\xi_2 = 0$ then the bottom terms must be zero, but it is 1. Ie, there is no choice of the vectors $|\chi\rangle$ and $|\xi\rangle$ which can make these equal to the vector $|1;3\rangle \otimes |1;3\rangle + |-1;3\rangle \otimes |-1;3\rangle$. This vector is NOT a product vector. In quantum mechanics it is called an entangled state.

4.Show that if I take

$$H = \frac{E}{2}\Sigma_1 \tag{20}$$

that in the Heisenberg representation, or any multiple of Σ_1 does not change in time. Show that the matrix

$$\Sigma_x = cos(\frac{E}{\hbar}t)\Sigma_2 - sin(\frac{E}{\hbar}t)\Sigma_3 \tag{21}$$

solves the equation of motion for Σ_x.

$$i\hbar\frac{\partial\Sigma_x}{\partial t} = [\Sigma_x, H] \tag{22}$$

(You will need to remember what the value is for the product of two Σ matrices)

[1– for recalling these relations]

$$\begin{aligned}\Sigma_1\Sigma_3 &= -\Sigma_3\Sigma_1 = -i\Sigma_2 \\ \Sigma_2\Sigma_1 &= -\Sigma_1\Sigma_2 = -i\Sigma_3\end{aligned} \tag{23}$$

Thus we have for a multiple of the Σ_1 matrix, eg, $\beta\Sigma_1$that

[1]

$$[\beta\Sigma_1, H] = \beta\frac{E}{2}(\Sigma_1\Sigma_1 - Sigma_1\Sigma_1 = 0 \tag{24}$$

Thus the right side of the Heisenberg equations is zero. Thus the time derivate of $\beta\Sigma_1$ is zero.

[1]

$$\frac{d\Sigma_x}{dt} = -\frac{E}{\hbar}sin(\frac{E}{\hbar}t)\Sigma_2 + \frac{E}{\hbar}cos(\frac{E}{\hbar}t)\Sigma_1 \tag{25}$$

and

[1]

$$\begin{aligned}[\Sigma_x, H] &= (cos(\frac{E}{\hbar}t)\Sigma_2 + sin(\frac{E}{\hbar}t)\Sigma_3)\frac{E}{2}\Sigma_1 - \frac{E}{2}\Sigma_1(cos(\frac{E}{\hbar}t)\Sigma_2 + sin(\frac{E}{\hbar}t)\Sigma_3) \\ &= \frac{E}{2}\left(cos(\frac{E}{\hbar}t)(-i)\Sigma_3 + sin(\frac{E}{\hbar}t)i\Sigma_2) - (cos(\frac{E}{\hbar}t)i\Sigma_3 + sin(\frac{E}{\hbar}t)(-i)\Sigma_2)\right) \\ &= iE\left(sin(\frac{E}{\hbar}t)\Sigma_2 - cos(\frac{E}{\hbar}t)\Sigma_3\right)\end{aligned} \tag{26}$$

Thus,

[1– ie for getting the answer.]

$$i\hbar\frac{d\Sigma_x}{dt} = [\Sigma_x, H] \tag{27}$$

as required.

Similarly if we define $\Sigma_y = cos(\frac{E}{\hbar}t)\Sigma_3 - sin(\frac{E}{\hbar}t)\Sigma_2$, it also oveys the Heisenberg equations of motion. Thus these two attributes act as though they rotate about the "z" axis defined by Σ_1.

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5. Show that if $|\psi\rangle$ is a normalised vector which depends on time, that

$$\langle\psi||\dot{\psi}\rangle + \langle\dot{\psi}||\psi\rangle = 0 \tag{28}$$

where the dot denotes a time derivative. By taking the time derivative of

$$\langle a|A|a\rangle = a \tag{29}$$

for an eigenvector $|a\rangle$ show that if A obeys the Heisenberg equations of motion that a does not change in time. (Note that in general $|a\rangle$ will change in time if A does.

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[1] Product rule of derivatives.

$$0 = \frac{d1}{dt} = \frac{d\langle a||a\rangle}{dt} = \frac{d\langle a|}{dt}|a\rangle + \langle a|\frac{d|a\rangle}{dt} \tag{30}$$

Since the definition of $\dot{}$ is just the time derivative, we have the proof.

Now,

$$\langle a|A|a\rangle = \langle a|a|a\rangle = a\langle a||a\rangle = a \tag{31}$$

Thus

[2– Key concepts– $A|a\rangle = a|\rangle$ amd the adjoint of this $\langle a|A = a\langle a|$; and Heisenberg equation for $\dot{A}$]

$$\frac{da}{dt} = \frac{d\langle a|A|a\rangle}{dt} = \frac{d\langle a|}{dt}A|a\rangle + \langle a|A\frac{d|a\rangle}{dt} + \langle a|\frac{dA}{dt}|a\rangle \tag{32}$$

$$= \frac{d\langle a|}{dt}a|a\rangle + \langle a|a\frac{d|a\rangle}{dt} + \langle a|\frac{dA}{dt}|a\rangle \tag{33}$$

$$= a\left(\frac{d\langle a|}{dt}|a\rangle + \langle a|\frac{d|a\rangle}{dt}\right) - \frac{i}{\hbar}\langle a|(AH - HA)|a\rangle \tag{34}$$

$$= 0 - \frac{i}{\hbar}\langle a|(aH - Ha)|a\rangle = -a\frac{i}{\hbar}(\langle a|(H - H)|a\rangle) = 0 \tag{35}$$

(recalling that if $A|a\rangle = a|a\rangle$ then $\langle a|A = a\langle a|$.) Ie, the eigenvalues of any matrix stay the same under the Heisenberg equations of motion of the attribute.

Ie, the new matrix representing the attribute has the same set of possible values as the old matrix at all times. This is what one would want.
