Physics 200-05
Assignment 7 Solutions

1. Consider the state vector

$$|\psi\rangle = \frac{1}{\sqrt{(2)}} \begin{pmatrix} 1 \\ \frac{1+i}{\sqrt{2}} \end{pmatrix} \tag{1}$$

a)What is the unit vector $|\phi\rangle$ orthogonal to this vector? Ie, $\langle\phi||\psi\rangle = 0$?

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[1]

$$|\phi\rangle = \frac{1}{\sqrt{(2)}} \begin{pmatrix} \frac{1+i}{\sqrt{2}} \\ -1 \end{pmatrix} \tag{2}$$

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b) Show that the matrix $A = |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|$ has eigenvalues ± 1 and eigenvectors $|\psi\rangle$ and $|\phi\rangle$. (Remember that $|\mu\rangle\langle\nu|$ is the product of a column vector times a row matrix, which is a 2x2 matrix if the $|\mu\rangle$ and $|\nu\rangle$ are 1x2 vectors.)

$$\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix} = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix} \tag{3}$$

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[2]

$$(|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|)|\psi\rangle = |\psi\rangle\langle\psi||\psi\rangle - |\phi\rangle\langle\phi|\psi \tag{4}$$
$$= |\psi\rangle(1) - |\phi\rangle(0) = |\psi\rangle \tag{5}$$

Ie, ψ is an eigenvector of A with eigenvalue 1. Similarly

$$(|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|)|\phi\rangle = |\psi\rangle\langle\psi||\phi\rangle - |\phi\rangle\langle\phi|\phi \tag{6}$$
$$|\psi\rangle(0) - |\phi\rangle(1) = (-1)|\phi\rangle \tag{7}$$

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2)Given that the probabilities for rain is
1cm $\rightarrow$ .5
2cm $\rightarrow$ .3
3cm $\rightarrow$ .1
0cm $\rightarrow$ .1

a) What is the expectation value for the rain amount? What is the uncertainty.

[2]

$$< rain >= \sum_i prob(rain_i)rain_i = .5(1) + .3(2) + .1(3) + .1(0) =$$

$$\Delta(rain)^2 = \sum_i prob(rain_i)(rain_i - < rain >)^2 = .5(1-1.4)^2 + .3(.2-1.4)^2 + .1(3-1.4)^2 + .1(0 -$$

$$= .5(.16) + .3(.36) + .1(2.56) + .1(1.96) =$$

Thus $\Delta(rain) = .8$.

a) If there were 30 days on which the above were the forcast, on how many days would expect there to at least 2 cm of rain?

[2]

At least 2 cm means from the list either 2 cm or 3 cm. The probability of one of these is .3+.1=.4 Thus the probability of rain on any day is .4. One would thus expect in 30 days that .4(30)=12 days of rain of at least 2 cm

3) Show that

$$[A, BC] = [A, B]C + B[A, C] \tag{11}$$

where $A,\ B,\ C$ are matrices and $[A, B] = AB - BA$ is the commutator.

[1]

$$[A, B]C + B[A, C] = (AB - BA)C + B(AC - CA) \tag{12}$$

$$= ABC - BAC + BAC - BCA = ABC - BCA = [A, BC] \tag{13}$$

Show that if X and P obey

$$[X, P] = i\hbar I \tag{14}$$

and if we define the Energy as

$$H = \frac{1}{2m}P^2 + \frac{k}{2}X^2 \tag{15}$$

where m and k are real numbers. Then

$$[X,H] = i\hbar\frac{1}{m}P \tag{16}$$

and

$$[P,H] = -i\hbar kX \tag{17}$$

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[2]

$$[X,H] = XH - HX = X(\frac{1}{2m}P^2 + \frac{k}{2}X^2) - (\frac{1}{2m}P^2 + \frac{k}{2}X^2)X \tag{18}$$

$$= \frac{1}{2m}[X,P^2] = \frac{1}{2m}([X,P]P + P[X,P]) = \frac{i\hbar}{2}(IP + PI) = i\hbar\frac{1}{m}P \tag{19}$$

$$[P,H] = P(\frac{1}{2m}P^2 + \frac{k}{2}X^2) - (\frac{1}{2m}P^2 + \frac{k}{2}X^2)P \tag{20}$$

$$= \frac{k}{2}(PX^2 - X^2P) = \frac{k}{2}[P,X^2] = \frac{k}{2}([P,X]X + X[P,X]) \tag{21}$$

$$\tag{22}$$

But

$$[P,X] = PX - XP = -(XP - PX) = -[X,P] = -i\hbar I \tag{23}$$

and thus

$$[P,H] = -i\hbar kX \tag{24}$$

as required.

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4) Given

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \frac{1-i}{\sqrt{2}} \end{pmatrix} \tag{25}$$

find the expectation value and the uncertainty of the attribute represented by the matrix

$$\Sigma_1 + \Sigma_3 \tag{26}$$

$$\Sigma_1 + \Sigma_3 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{27}$$

and
[2]

$$< \Sigma_1 + \Sigma_3 >= \langle\psi|(\Sigma_1 + \Sigma_3)|\psi\rangle \tag{28}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \frac{1+i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \frac{1-i}{\sqrt{2}} \end{pmatrix} \tag{29}$$

$$= \begin{pmatrix} \frac{\sqrt{2}+1-i}{2} & \frac{\sqrt{2}-1+i}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1-i}{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \tag{30}$$

[1]

$$(\Sigma_1 + \Sigma_3)^2 = \Sigma_1^2 + \Sigma_1\Sigma_3 + \Sigma_3\Sigma_1 + \Sigma_3^2 \tag{31}$$

$$= 2I \tag{32}$$

and Thus
[1]

$$\langle\psi|(\Sigma_1 + \Sigma_3 - \frac{1}{\sqrt{2}}I)^2|\psi\rangle = \langle\psi| \left(\Sigma_1 + \Sigma_3)^2 - 2\frac{1}{\sqrt{2}}(\Sigma_1 + \Sigma_3) + \frac{1}{2} \right) |\psi\rangle \tag{33}$$

$$= 2 - \frac{1}{2} = \frac{3}{2} \tag{34}$$

and

$$\Delta(\Sigma_1 + \Sigma_3) = \sqrt{\frac{3}{2}} \tag{35}$$