Physics 200-05

Assignment 6

1) What is the derivative with respect to time of the following matrices? Are they Hermitean matrices? In case a) find the eigenvectors. Remember, i in all equations is treated as just another variable, except that $i^2 = -1$.

[**Note** – This question does not make clear whether I mean the original matrices or the derivatives. Thus either one could be used. However the student should state which]

a)

$$\begin{pmatrix} 1 & e^{i\omega t} \\ e^{-i\omega t} & 1 \end{pmatrix} \tag{1}$$

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[1]

$$\begin{pmatrix} 0 & i\omega e^{i\omega t} \\ -i\omega e^{-i\omega t} & 0 \end{pmatrix} \tag{2}$$

Ie, you differentiate a matrix by differentiating each entry.

$$\frac{d}{dt}A = lim_{\epsilon\to 0}\frac{A(t+\epsilon)-A(t)}{\epsilon} \tag{3}$$

and since addition and subtraction are element by element and since multiplication (division) is also of each element, this leads to each entry being taken the derivative of.

[1]

This matrix is Hermitean although the derivative is not. $(e^{i\omega t})^* = e^{-i\omega t}$

The eigenvectors of the original matrix are:

$$A = \begin{pmatrix} 1 & e^{i\omega t} \\ e^{-i\omega t} & 1 \end{pmatrix} = I + cos(\omega t)\Sigma_1 - \sin(\omega t)\Sigma_2 \tag{4}$$

and thus the eigenvalues are

[1]

$$a = 1 \pm \sqrt{cos(\omega t)^2 + \sin(\omega t)^2} = \{2, 0\} \tag{5}$$

For the case of the eigenvalue 0, the eigenvector is

[1] (Half for vector and half for normalisation)

$$\begin{pmatrix} 1 & e^{i\omega t} \\ e^{-i\omega t} & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \tag{6}$$

or

$$b = -e^{-i\omega t} a \tag{7}$$

or

$$|0\rangle = a \begin{pmatrix} 1 \\ -e^{-i\omega t} \end{pmatrix} \tag{8}$$

To normalise this, we take

$$1 = \langle 0||0\rangle = a^* \begin{pmatrix} 1 & -e^{+i\omega t} \end{pmatrix} a \begin{pmatrix} 1 \\ -e^{-i\omega t} \end{pmatrix} = 2a^* a \tag{9}$$

and thus we want $a = 1/\sqrt{2}$.

[2]

We can go through the same for the other eigenvector, however we can make it easier. The other eigenvector is orthogonal to this one. (the eigenvectors of a Hermitean matrix are orthogonal to each other). Ie, $\langle 0||2\rangle = 0$ from which we immediately see that

$$|2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\omega t} \\ 1 \end{pmatrix} \tag{10}$$

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b)

$$\begin{pmatrix} \cos(\omega t) & i\sin(\omega t) \\ i\sin(\omega t) & \cos(\omega t) \end{pmatrix} \tag{11}$$

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[1]

$$\begin{pmatrix} -\omega\sin(\omega t) & i\omega\cos(\omega t) \\ i\omega\cos(\omega t) & -\omega\sin(\omega t) \end{pmatrix} \tag{12}$$

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2) What are the eigenvalues of the following matrices?
a)

$$\begin{pmatrix} 1 & 3i \\ -3i & -1 \end{pmatrix} \tag{13}$$

$$\begin{pmatrix} 1 & 3i \\ -3i & -1 \end{pmatrix} = -3\Sigma_2 + \Sigma_3 \tag{14}$$

[1]
Thus the eigenvalues are $\pm\sqrt{(-3)^2 + 1^2} = \pm\sqrt{10}$.

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b)

$$\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \tag{15}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} = \frac{1}{2}I + 2\Sigma_1 + \frac{1}{2}\Sigma_3 \tag{16}$$

[1]
so the eigenvalues are $\frac{1}{2} \pm \sqrt{2^2 + (\frac{1}{2})^2} = \frac{1}{2}(1 \pm \sqrt{17})$.

3) a) Given the expression

$$A = \beta_0 I + \vec{\beta} \cdot \vec{\Sigma} \tag{17}$$

Show that the matrix

$$\frac{1}{\beta_0^2 - \vec{\beta} \cdot \vec{\beta}} \left(\beta_0 I - \vec{\beta} \cdot \vec{\Sigma}\right) \tag{18}$$

is the inverse of A. What is the condition that the matrix A not have an inverse.

[1] for recognizing what $\vec{\beta} \cdot \vec{\Sigma}$ means. [1] for recognizing that $\Sigma_i \Sigma_i = I$ and [1] for $\Sigma_i \Sigma_j + \Sigma_j \Sigma_i = 0$ for all $i \neq j$. Finally [1] for normalisation term and [1] for recognizing that if $(\beta_0^2 - \vec{\beta} \cdot \vec{\beta}) = 0$, then no inverse exists.

$$\left(\beta_0 I - \vec{\beta}\cdot\vec{\Sigma}\right)(\beta_0 I + \vec{\beta}\cdot\vec{\Sigma}) \tag{19}$$

$$= \beta_0^2 I * I + \beta_0\vec{\beta}\cdot(I\vec{\Sigma}) - \beta_0\vec{\beta}\cdot(\vec{\Sigma}I - \vec{\beta}\cdot\vec{\Sigma}\vec{\beta}\cdot\vec{\Sigma}) \tag{20}$$

$$= \beta_0^2 I * I - \vec{\beta}\cdot\vec{\beta}I = (\beta_0^2 - \vec{\beta}\cdot\vec{\beta})I \tag{21}$$

And thus if we divide by $\beta_0^2 - \vec{\beta}\cdot\vec{\beta}$ we get the identity matrix. Thus thatmatrix is the inverse of A (unless of course $\beta_0^2 - \vec{\beta}\cdot\vec{\beta} = 0$ in which case A has no inverse.

b) What is the inverse of the two matrices in problem 2

[1] each for finding β for these and [1] each for finding the inverse.

$$\begin{pmatrix} 1 & 3i \\ -3i & -1 \end{pmatrix} = -3\Sigma_2 + \Sigma_3 \tag{22}$$

Thus the inverse is

$$\frac{1}{-((-3)^2 + 1^2)}(3\Sigma_2 - \Sigma_3) = \frac{1}{10}\begin{pmatrix} -1 & -3i \\ 3i & 1 \end{pmatrix} \tag{23}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} = \frac{1}{2}I + 2\Sigma_1 + \frac{1}{2}\Sigma_3 \tag{24}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} = \frac{1}{2}I + 2\Sigma_1 + \frac{1}{2}\Sigma_3 \tag{25}$$

And thus the inverse would be

$$\frac{1}{\frac{1}{4} - 4 - \frac{1}{4}}(\frac{1}{2}I - 2\Sigma_1 - \frac{1}{2}\Sigma_3) = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{pmatrix} \tag{26}$$

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4) A particle is found by measurement to have the the value +1 for the physical attribute represented by the matrix Σ_1. What is the probability that if the physical attribute represented by the $\Sigma_1 + \Sigma_2$ matrix is measured, its value is found to be the largest eigenvalue.

[1] for recognizing what the first part says about the state of the system, [1] for the eigenvalues of $\Sigma_1 + \Sigma_2$, and [1] for the largest eigenvector. [1] for the amplitude (either as below or some other method) and [1] for the probability.

The eigenvector for the Σ_1 matrix for the +1 eigenvalue is

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

The largest eigenvalue of $\Sigma_1 + \Sigma_2$ is $+\sqrt{1^2+1^2} = \sqrt{2}$ The eigenvector is

$$(\Sigma_1 + \Sigma_2)|\sqrt{2}\rangle = \sqrt{2}|\sqrt{2}\rangle \tag{27}$$

or

$$\begin{pmatrix}0 & 1-i\\ 1+i & 0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \sqrt{2}\begin{pmatrix}a\\b\end{pmatrix} \tag{28}$$

or

$$(1-i)b = \sqrt{2}a$$

and

$$|\sqrt{2}\rangle = b\begin{pmatrix}\frac{1-i}{\sqrt{2}}\\1\end{pmatrix}$$

or after normalisation

$$|\sqrt{2}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}\frac{1-i}{\sqrt{2}}\\1\end{pmatrix}$$

The probability if the state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

is

$$Prob = |\langle\sqrt{2}||\psi\rangle|^2 = |\frac{1}{\sqrt{4}}(\frac{1+i}{\sqrt{2}}+1)|^2 = \frac{2+\sqrt{2}}{4}$$

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5) A particle is found by measurement to have the value of +1 for the attribute represented by Σ_3. Then the attribute Σ_2 is measured and also found to have value +1. What is the probability that if Σ_3 is remeasured, its value is found to be -1?

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[1] for realising that after the second measurement the state is the eigenvector is Σ_2 no matter what the initial state was. and [1] for finding the probability of Σ_3.

After the measurement of Σ_2, the fact that Σ_3 was first measured and found to be $+1$ is irrelevant. Each measurement replaces the previous one. Thus the probability is $\frac{1}{2}$.