Physics 200-04
Assignment 3

1. Aberration: Assume that a star is seen to be at an angle of θ with the positive x axis and assume that its location is in the $x-y$ plane. Show that in a frame travelling at velocity v with repect to the first frame along the x axis the angle $\tilde{\theta}$ is given by

$$tan(\tilde{\theta}) = \frac{sin(\theta)}{\cos(\theta)+v}\sqrt{1-v^2} \tag{1}$$

Show that to lowest order in v, this is the same as the expression for the angle as for the non-relativistic aberration angle.

As a spaceship travels at very near the velocity of light, show that the stars all become crowded into the forward direction.

The light is travelling from the star in direction θ toward the observer. The components of w_x and w_y for the light are

$$w_x = -cos(\theta) \tag{2}$$
$$w_y = -sin(\theta) \tag{3}$$

since the velocity is $c=1$ and the direction of the light is from the star to the observer. In the new frame moving with velocity The change in w_x and w_y in teh new frame is (using the change of velocities)

$$\tilde{w}_x = \frac{(w_x - v)}{1 - w_x v} \tag{4}$$
$$\tilde{w}_y = \sqrt{1-v^2}\frac{w_y}{1 - w_x v} \tag{5}$$

This means that the tangent of the angle in the new frame is

$$\begin{aligned} \tan(\tilde{\theta}) &= \frac{\tilde{w}_y}{w_x} = \sqrt{1-v^2}\frac{w_y}{w_x - v} = \sqrt{1-v^2}\frac{-sin(\theta)}{-cos(\theta)-v} && (6) \\ &= \sqrt{1-v^2}\frac{sin(\theta)}{\cos(\theta)+v} && (7) \end{aligned}$$

To lowest order in v(Ie, taking a taylor series and keeping only terms up to linear in v) we have

$$\frac{\tan(\tilde{\theta}) \approx \frac{sin(\theta)}{\cos(\theta)+v} \approx tan(\theta)(1-v}{cos(\theta)} \tag{8}$$

which is exactly the non-relativistic expression for the aberration.

We note that for all θ less than $\pi/2$, the expression for $\tan(\tilde{\theta})$ is smaller than it would be if $v = 0$. Since $\tan(\tilde{\theta})$ is monotonic in $\tilde{\theta}$, the smaller size means that the angle $\tilde{\theta}$ would be smaller than it would be if $v = 0$. Ie the stars are further into the forward direction. This same is true in the backward direction, where $\cos(\theta) + v < 0$, where the expression is more negative than for $v = 0$. Ie all angles are smaller– the star images are displaced in teh forward direction. If v is very near 1, then that deflection of the light is large and almost all the angles are pushed into the forward direction (all stars which obey $cos(\theta) > -v$ are pushed into the forward half hemisphere.

(Ie, in star treck and star wars, as the space ship nears the speed of light, the stars do not whoosh behind the starship, but all move in front of the starship).

Note that if θ is very small (so that $sin(\theta) \approx \theta$ and $\cos(\theta) \approx 1$, we have

$$\tilde{\theta} \approx \theta\sqrt{\frac{1-v}{1+v}} \tag{9}$$

2.Twins Paradox: In class I gave one explanation of the twin's paradox. This will be another one. (It was one that I published in the American Journal of Physics in the early 1980's).

Two twins, Alice and Bob take trips. Alice's trip is to stay at home. Bob travels away at high velocity v, for a time T in Alice's frame, and immediately returns home with the same velocity. They both have telescopes to look at the other. Before they left each carefully measured the other's height (in a direction perpendicular to the direction of travel). Each carefully observes the other during the whole trip. By measuring the angular height of the other as seen through the telescope, each can determine the distance that that the other is away from himself by using

$$distance = \frac{H}{\delta\theta} \tag{10}$$

where H is the known height of the other and $\delta\theta$ is the measured height of the other as seen through the telescope. Assuming that the velocity of light is c they can thus calculate how long before he sees the other, the light left the other's position.

a) Show that all along the trip each "sees" (after correcting for the time of travel) the other's clock tick more slowly by $\sqrt{1-v^2}$.

b) On a graph, plot the path (distance away vs time as corrected for light travel time) that the person sees the other travel. Ie, if

$$X(t) = \frac{\delta\theta(t)}{H} \tag{11}$$

where t is the time at which that angle $\delta\theta$ is measured by that observer, plot $t - X(t)$ vs $X(t)$ for each observer seeing the other.

Note you will probably need to use the aberration formula from problem set 1 where you can assume that *theta* is very small. (Ie, the height over the distance is small)

In particular see what happens to the height as measured by B at his turnaround of A, and what this means for the distance he ascribes to A and the time that the light left A.

Let us first analyse it from A's point of view. The light from B at position x takes x to reach A. Thus, at time t in A's frame B will be at point vt and the light from him will reach A at time $\tau = t + vt = (1+v)t$ B's clock will read a time of $\sqrt{1-v^2}t$ at the time the light leaves. Thus, at time T, A will see B's clock reading $\sqrt{1-v^2}t = \sqrt{\frac{1-v}{1+V}}\tau$. A will also B's height as

$$\delta\theta = \frac{H}{vt} = H\frac{(1+v)}{v\tau} \tag{12}$$

Thus at time τ, A will ascribe the distance $\frac{X_B=H}{\delta\theta=\frac{v\tau}{1+v}}$ to B and the time $\tau_B = \tau - X = \frac{\tau}{1+v}$ to B. She will see that $B's$ clock at that time reads $\sqrt{\frac{1-v}{1+v}}\tau = \sqrt{1-v^2}\tau_B$.

We note that τ is exactly the time t as measured in A's frame. Ie, this is for A a consitant way of determining the time and location of B.

We can carry out the same analysis for the times $t > T$ after B has turned around. at the time $t > T$, B will be at $v(2T-t)$. The light will take $v(2T-t)$ to reach A, and A's clock will read $\tau = t + v(2T-t)$ when the light reaches A. A will see the angular height of B to be $\frac{\delta\theta=H}{v(2T-t)=H\frac{1-v}{2T-\tau}}$. Again, using this angular height to determine how far away B is and to ascribe the time to B corrected for the light travel time from b to A as determined by the angular height, one agains finds that B just follows the expected path in A's frame. Ie, using this technique, A will simply ascribe to B the position and time of A's frame. Similarly all along the path B's clock will run $\sqrt{1-v^2}$ slower than A's time.

For B the crucial idea is the above aberation formula, and in particular the change in aberration when B turns around.

Let us look at the light travelling from A to B. Let us look at the lightwhich leaves A at time t. It has to catch up to B who is travelling at velocity v. The intersection will occur when $c(t'-t) = vt'$ or $t' = \frac{t}{1-v}$. If one had an observer in A's frame at that point and that time, that observer would see A to have angular height of

$$\delta\theta = \frac{H}{vt'} = \frac{H1-v}{t} \tag{13}$$

But for B, the angular height is changed by the aberration formula from problem 1. Since the angle is small, we have (recalling that during the first part of the trip B is travelling away from A), we have

$$\delta\tilde{\theta} = = \sqrt{\frac{1+v}{1-v}}\delta\theta = \sqrt{\frac{1+v}{1-v}}\frac{H}{vt'} \tag{14}$$

$$= \frac{H}{v\frac{\tau}{\sqrt{1-v^2}}} \tag{15}$$

where $\tau = t'\sqrt{1-v^2}$ is the reading on B's clock. Thus at time τ B will see the clock on A reading $t = t'(1-v) = \tau\sqrt{\frac{1-v}{1+v}}$. At this time B will regard A as being at a distance $X_A = \frac{H}{\delta\tilde{\theta}} = v\tau$ away. Ie, B will see A at its time τ in exactly the same way as A sees B at the same time τ on A's clock. In this part of the trip, everything looks symmetric with respect to A and B.

However, when B gets to teh turn around point at his time $\tau = \sqrt{1-v^2}T$, things suddenly change. His velocity reverses, and more particularly, the aberration reverses. Suddenly at time $\tau = \sqrt{1-v^2}T$ when B sees the time $t = (1-v)T$ on A's clock, The aberration switches from

$$\delta\tilde{\theta} = \sqrt{\frac{1+v}{1-v}}\delta\theta = \sqrt{\frac{1+v}{1-v}}\frac{H}{vT} \tag{16}$$

to

$$\delta\tilde{\theta} = \sqrt{\frac{1-v}{1+v}}\delta\theta = \sqrt{\frac{1-v}{1+v}}\frac{H}{vT} \tag{17}$$

Ie, A suddenly looks smaller. By the way B defines how far away A is, this also means that A suddenly gets further away. Furthermore, since A is further away, the light has taken a longer time to reach B, and the time when the light left A is further in the past than it was just before B turned around. Ie, B suddenly sees A as "instantly" becoming further away, and thus also further into the past. Thus the path that B ascribes to A is that A travels away from B with velocity v until the time $T\sqrt{\frac{1-v}{1+v}}$. At that time, A "instantly"(ie without the time showing on A's clock changing) is seen by B to be at the location $v\sqrt{\frac{1+v}{1-v}}T$, and at the time $(1-2v)T\sqrt{\frac{1+v}{1-v}}$. Note that this time will be less than zero (ie according to B, A will suddenly be at a time earlier than the time at which A left B if $v > \frac{1}{2}$.) From now on A will travel back toward B with velocity v, and again with a slower clock than B's time.

Ie, B will always deduce that A's clock ticks slower than his own time does, but that A suddenly (when B turns around) makes a trip back in time but just enough so that the total time which is seen on A's clock is 2T while B's is $\sqrt{1-v^2}2T$.

3.Doppler shift: A light flashes once a second (according to its own clock). It is travelling with velocity .9c with respect to an observer, in the direction of the positive x direction and at a distance of 1 light day away along the y direction. What is the frequency of the flashes as seen by the observer as a function of time. What is the limiting frequency when the x value is very large and negative, when $x = 0$ and when x is large and positive.

If the light is emitted at time t, the time it takes for the light to reach the origin is $\sqrt{y^2 + (vt)^2}$ and will thus arrive at the origin at the time $\tau = t + \sqrt{y^2 + (vt)^2}$ The time as measured in the moving objects frame is $\tilde{t} = \sqrt{1 - v^2}t$ Thus the relation between the time at which the light arrives at the origin and the time at which it was emitted is

$$\tau = \frac{1}{\sqrt{1 - v^2}}\tilde{t} + \sqrt{y^2 + \frac{v^2}{1 - v^2}\tilde{t}^2} \tag{18}$$

The time between the reception of the pulses is

$$\Delta\tau = \frac{1}{\sqrt{1 - v^2}}\Delta\tilde{t} + \frac{1}{\sqrt{y^2 + \frac{v^2}{1-v^2}\tilde{t}^2}}\frac{v^2}{1 - v^2}\tilde{t}\Delta\tilde{t} \tag{19}$$

For large negative time $\tilde{t}$, the y^2 term in the square root is negligible and

$$\Delta\tau = \frac{1 - v}{\sqrt{1 - v^2}}\Delta\tilde{t} = \sqrt{\frac{1 - v}{1 + v}} \tag{20}$$

Ie, the received frequency is higher than the frequency of the source.

At a time $\tilde{t} = 0$ when the moving object is moving transversly to the observer, the expression becomes

$$\Delta\tau = \frac{\Delta t}{\sqrt{1 - v^2}} \tag{21}$$

This is called the transverse doppler shift. The shift in the pulse frequency is purely that due to the different rates at which the moving and stationary clock tick. Finally if the clock is at very large positive time $\tilde{t}$, the expression is

$$\Delta\tau = \sqrt{\frac{1 + v}{1 - v}}\Delta\tilde{t} \tag{22}$$

which is larger than the ticks of the source.

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4. An hydrogen atom of rest mass energy 1Gev emits a photon with energy 10 ev. What is the rest mass of the H atom after the emission? If the photon had an energy of .5GeV, what would the rest mass energy of the H atom be afterward.

This is just a specific application of the problem done in class. There we found that

$$m^2 = M^2 - 2\epsilon M \tag{23}$$

was the relation between the masses. Since we are measuring the masses and teh energy in the same units in this problem, the two possibilities are

$$m^2 = (1GeV)^2 - 2(10ev)(1GeV) = 10^{18}ev - 10(10^9 eV) = .99999999 10^18 GeV \tag{24}$$

or

$$m = \sqrt{M^2 - 2\epsilon M} \approx M - \epsilon \tag{25}$$

which means that the mass of the resultant particle is just 10ev smaller than 1GeV.

If the energy is .5GeV however, the mass of the final partile is

$$m^2 = M^2 - 2\epsilon M = 0 \tag{26}$$

since $2\epsilon = M$. Ie, the other particle must also be a photon.

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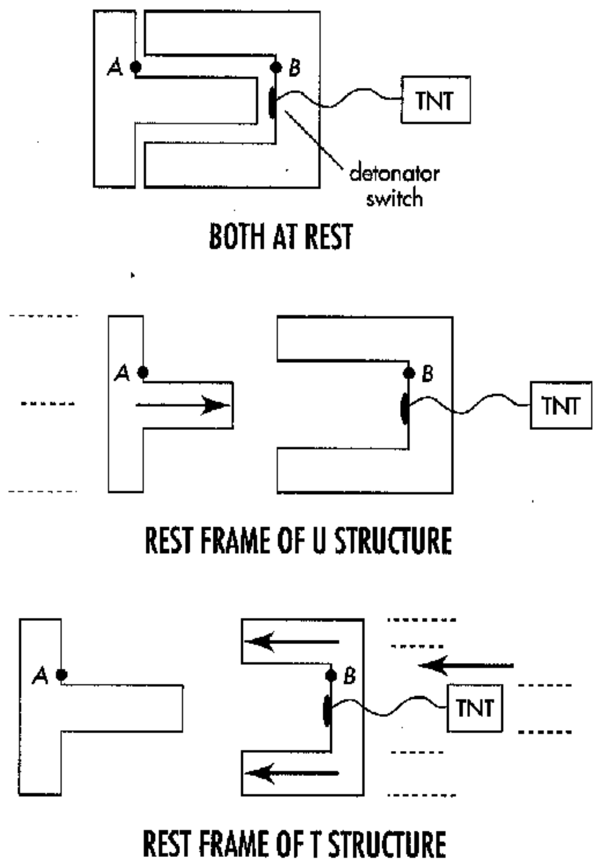
5. This problem is taken from Wheeler and Taylor (6.5)

A structure as in figure 1 is such that the projection of the structure A would just touch the detonator when the flanges of figure A just touch the bars of Figure B. Now A is shot at B at a sizeable fraction of the speed of light. In the frame of figure B, A is shortened and thus the flanges will touch the bars of A before the projection touches the detonator, and will stop A befor the projection hits the detonator. In A's frame however, B is shorter, and the projection of A hits the detonator before the flanges touch B and the bomb is detonated.

Is the bomb detonated or not? What is wrong with the argument that predicts the opposite?

This is precisely the same as the pole in the barn problem. We see that depending on which frame we look at the problem in, either the front of the T hits the detonator before the flanges hit the ends of the U, or the flanges hit first. But this means that the flanges hitting and the end of the T hitting the detonator are spacelike separated. Thus no signal can travel from one of those events to the other. In the frame of the U, the flanges hit first. But no signal can be sent from teh flanges to the end of the T, and thus nothing can stop the end of the T before it hits the detonator. In this frame the flanges will hit the ends of the U, and a shock wave will travel down the T, but the end of the T will keep moving and hit the detonator long before the shock wave can reach the end of the T. The T cannot stip instantly.

The detonator will go off.



EXERCISE 6-5. ***Both at rest:*** *The leg of the T almost reaches the detonator switch when both the T and the U are at rest. Points* A *and* B *are used in part b of the exercise.* ***Rest frame of U structure:*** *The leg of the moving T is Lorentz contracted in the rest frame of the U. Does this mean that the explosion will not take place?* ***Rest frame of T structure:*** *The legs of the moving U are Lorentz-contracted in the rest frame of the T. Does this mean explosion will take place?*