Physics 200-05
Assignment 2

1. Give the two matricees

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 1 & 0 \end{pmatrix} \tag{1}$$

$$B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

Find the matricees AB, BA, A^T,B^T, and B^{-1}. Does the matrix A represent a rotation? Does the matrix B?

(Recall that a rotation is defined by the requirement that $A^T I A = I$. Ie, the identity matrix plays the roll of the metric G.

Solution:

$$AB = \begin{pmatrix} 2 & -1 & 3 \\ 4 & -2 & 1 \\ 1 & -3 & 0 \end{pmatrix} \tag{3}$$

$$BA = \begin{pmatrix} -2 & -4 & -1 \\ 1 & 2 & 3 \\ 3 & 1 & 0 \end{pmatrix} \tag{4}$$

$$A^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 1 & 0 \end{pmatrix} \tag{5}$$

$$B^T = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6}$$

$$B^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7}$$

$$\tag{8}$$

The matrix B is a rotation since $B^T = B^{-1}$. The matrix A is not since

$$A^T A = \begin{pmatrix} 14 & 13 & 5 \\ 13 & 21 & 10 \\ 5 & 10 & 10 \end{pmatrix} \neq I \tag{9}$$

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2. i) Show that

$$\sinh(\theta)\cosh(\theta') + \cosh(\theta)\sinh(\theta') = \sinh(\theta + \theta') \tag{10}$$

$$\cosh(\theta)\cosh(\theta') + \sinh(\theta)\sinh(\theta') = \cosh(\theta + \theta') \tag{11}$$

Note the similarities **and** differences with the trigonometric formulas you are (I hope) more familiar with.

ii) Using the above formula show that two successive Lorentz transformations both along the x direction are such that if the velocity of transformation from the first to the second frame is v_1 and of the second to the third frame is v_2 then the velocity from the first to third frame is

$$\frac{v_f}{c} = \frac{\frac{v_1}{c} + \frac{v_2}{c}}{1 + \frac{v_1 v_2}{c^2}} \tag{12}$$

(Use the fact that $\tanh(\theta) = \frac{v}{c}$). Ie, while the rapidities (the "angle" in the hyperbolic function representation of the Lorentz transformations) of successive Lorentz transformations add, the velocities do not.

$$\sinh(\theta)\cosh(\theta') + \cosh(\theta)\sinh(\theta') = \tag{13}$$

$$\frac{1}{4}\left(e^{\theta} - e^{-}\theta)(e^{\theta'} + e^{-\theta'}\right) + \frac{1}{4}\left(e^{\theta} + e^{-\theta})(e^{\theta'} - e^{-\theta'}\right) = \tag{14}$$

$$= \frac{2}{4}\left(e^{\theta+\theta'} - e^{-\theta-\theta'}\right) \tag{15}$$

$$= sinh(\theta + \theta') \tag{16}$$

$$\cosh(\theta)\cosh(\theta') + \sinh(\theta)\sinh(\theta') = \tag{17}$$

$$\frac{1}{4}\left(e^{\theta} + e^{-\theta})(e^{\theta'} + e^{-\theta'}\right) + \frac{1}{4}\left(e^{\theta} - e^{-\theta})(e^{\theta'} - e^{-\theta'}\right) = \tag{18}$$

$$= \frac{2}{4}\left(e^{\theta+\theta'} + e^{-\theta-\theta'}\right) \tag{19}$$

$$= cosh(\theta + \theta') \tag{20}$$

Let us take $c = 1$.

$$\tilde{t} = cosh(\theta)t - sinh(\theta)x \tag{21}$$

$$\tilde{x} = cosh(\theta)x - sinh(\theta)t \tag{22}$$

$$\tag{23}$$

and

$$\tilde{\tilde{t}} = cosh(\theta')\tilde{t} - sinh(\theta')\tilde{x} \tag{24}$$

$$\tilde{\tilde{x}} = cosh(\theta')\tilde{x} - sinh(\theta')\tilde{t} \tag{25}$$

Substituting the first set of equations into the second we get

$$\tilde{\tilde{t}} = cosh(\theta')(cosh(\theta)t - sinh(\theta)x) - sinh(\theta')(cosh(\theta)x - sinh(\theta)t) \quad (26)$$
$$\tilde{\tilde{x}} = cosh(\theta')(cosh(\theta)x - sinh(\theta)t) - sinh(\theta')(cosh(\theta)t - sinh(\theta)x) \quad (27)$$

or

$$\tilde{\tilde{t}} = (cosh(\theta')cosh(\theta) + sinh(\theta')sinh(\theta))t - (cosh(\theta')sinh(\theta) + sinh(\theta')cosh(\theta))x \quad (28)$$
$$= cosh(\theta+\theta')t - sinh(\theta+\theta')x \quad (29)$$
$$\tilde{\tilde{x}} = (cosh(\theta')cosh(\theta) + sinh(\theta')sinh(\theta))x - (cosh(\theta')sinh(\theta) + sinh(\theta')cosh(\theta))t \quad (30)$$
$$= cosh(\theta+\theta')x - sinh(\theta+\theta')t \quad (31)$$

Now,

$$\frac{v_f}{c} = tanh(\theta+\theta') = \frac{sinh(\theta+\theta')}{\cosh(\theta+\theta')} \quad (32)$$
$$= \frac{cosh(\theta)sinh(\theta') + sinh(\theta)\cosh(\theta')}{cosh(\theta)cosh(\theta') + sinh(\theta)sinh(\theta')} \quad (33)$$
$$= \frac{tanh(\theta) + \tanh(\theta')}{1 + \tanh(\theta)tanh(\theta')} = \frac{v+v'}{1+vv'} \quad (34)$$

Note that the velocities do not add (the final velocity is not just the sum of the two velocities) even though the rapidities (θ and θ') do add to give the final rapidity.

3.) Do a Lorentz transformation in the x direction with $v_x = c/\sqrt{3}$. Now in the new frame do a Lorentz transformation in the y direction with the velocity $v_y = c/\sqrt{2}$. Show that the combined transformation is equivalent to a Lorentz transformation along the direction at 45^o between the x and y axes with a velocity of $\sqrt{2/3}c$ followed by a rotation around the z axis.

While the full set of Lorentz transformations form a group, the boosts (which are the Lorentz transformations which leave two of the space-like directions the same) do not. This observation eventually leads to what is called the Thomas Precession, and is one of the purely kinematic terms which governs the precession of the spins of electrons in an atom. Two boosts along non-parallel directions is equivalent to a Lorentz boost plus a rotation is a generic feature of Lorentz transformations.

The easiest way to do this one is to do it via matrices. Again let us take c=1. If $v = 1/\sqrt{3}$, then $\gamma = \sqrt{3/2}$. Thus the Lorentz transformation to this new frame is

$$x' = \sqrt{3/2}(x - t/\sqrt{3}) \quad (35)$$
$$t' = \sqrt{3/2}(t - x/\sqrt{3}) \quad (36)$$
$$y' = y \quad (37)$$

If $v = 1/\sqrt{2}$, then $\gamma = \sqrt{2}$ and the transformation from the prime to doubleprime frame is

$$x'' = x' = \sqrt{3/2}(x - t/\sqrt{3}) \tag{38}$$

$$t'' = \sqrt{2}(t' - y'/\sqrt{2}) = \sqrt{3}t - x - y \tag{39}$$

$$y'' = \sqrt{2}(y' - t'\frac{1}{\sqrt{2}}) = \sqrt{2}y + x - \sqrt{3/2}t \tag{40}$$

A transformation along the x=y axis by a velocity $\sqrt{2/3}$, with a $\gamma = \sqrt{3}$ gives a transformation of

$$\tilde{t} = \sqrt{3}(t - \sqrt{2/3}(x + y)/\sqrt{2}) = \sqrt{3}t - x - y \tag{41}$$

$$(\tilde{x} - \tilde{y})/\sqrt{2} = (x - y)/\sqrt{2} \tag{42}$$

$$(\tilde{x} + \tilde{y})/\sqrt{2} = \sqrt{3}(\frac{(x + y)}{\sqrt{2}} - \sqrt{2/3}t) \tag{43}$$

We thus see that $\tilde{t} = t''$. We can also write

$$\tilde{x} = \frac{1 + \sqrt{3}}{2}x + \frac{\sqrt{3} - 1}{2}y + \frac{t}{\sqrt{3}} \tag{44}$$

$$\tilde{y} = \frac{\sqrt{3} + 1}{2}y + \frac{\sqrt{3} - 1}{2}x - \frac{t}{\sqrt{3}} \tag{45}$$

Assuming that $x'' = cos(\theta)\tilde{x} + sin(\theta)\tilde{y}$, and similarly for y'' we require that

$$cos(\theta)\frac{\sqrt{3} - 1}{2} + sin(\theta)\frac{\sqrt{3} + 1}{2} = 0 \tag{46}$$

since x'' does not have any y in it. Thus we require $tan(\theta) = -\frac{\sqrt{3}-1}{\sqrt{3}+1}$. Checking the other terms, this rotation gives us complete agreement.

In simpler terms, because the times, $\tilde{t}$ and t'' are the same, and relation between the double primed and the tilde coordinates cannot involve the time in the transformation from one to the other. The also cannot involve z'' or $\tilde{z}$. Thus they can only involve $x''y''$. The only such transformations are rotations.

4.) Show the consistancy of the special relativistic formulas. Consider the following synchronization of clocks. Alice and her friend Amy get together at the origin and sychronize their clocks to each other, ensuring that both show exactly the same time and run at the same rate. Now Amy very slowly (with a velocity δv approaching zero) moves away from Alice to a location X along Alice's x axis. Show that in Alice's frame, the time on Amy's clock at X will be synchronized with her clock. (Show that in the limit as δv goes to zero, the time difference between Amy's time to get to the location X and Alice's time for Amy to get to X are the same.) Now let us look at this process from Bob's point of view, who is moving with velocity v with respect to Alice. Show that Bob will calculate the difference between Amy's time to get to X and Alice's time is $\frac{vX}{c^2\sqrt{1-\frac{v^2}{c^2}}}$.

Hint, use the expression for the rate of Alice's and Amy's clocks according to Bob (time dilation) and look at the difference to lowest order in δv. Find the time it takes Amy to get to the point X and express the time difference in terms of X.

For Alice, Amy's clock will tick at the rate

$$t_{amy} = \frac{1}{\sqrt{1-\delta v^2}} t_{Alice}. \tag{47}$$

For Amy to travel to the point X in Alice's frame, she will have to travel for a time $X/\delta v$. Thus in that time, Amy's time will differ from Alice's time by

$$t_{amy} - t_{alice} = \left(\frac{1}{\sqrt{1-\delta v^2}} - 1\right)\frac{X}{\delta v} \approx \frac{\delta v^2}{2}\frac{X}{\delta v} = \frac{1}{2}\delta v X \tag{48}$$

Ie, in the limit as δv goes to zero, the difference between Amy's time and alice's time goes to zero.

Now let us look at this problem from Bob's point of view. His time is related to Alice's time by

$$t_{bob-alice} = \frac{1}{\sqrt{1-v^2}} t_{alice} \tag{49}$$

$$t_{bob-amy} = \frac{1}{\sqrt{1-(v+\delta v)^2}} t_{amy} = \frac{1-\sqrt{1-\delta v^2}}{\sqrt{1-(v+\delta v)^2}} t_{alice} \tag{50}$$

but the trip takes a time of $t_{alice} = X/\delta v$. Thus for Bob, his time when Amy reaches the point X is

$$t_{bob-alice} = \frac{1}{\sqrt{1-v^2}}\frac{X}{\delta v} \tag{51}$$

$$t_{bob-amy} = \frac{1}{\sqrt{1-(v+\delta v)^2}} t_{amy} = \frac{1-\sqrt{1-\delta v^2}}{\sqrt{1-(v+\delta v)^2}} t_{alice} \tag{52}$$

$$\approx t_{bob-alice} + \frac{v\delta v}{\sqrt{1-v^2}}\frac{X}{\delta v} = t_{bob-alice} + \frac{vX}{\sqrt{1-v^2}} \tag{53}$$

Ie, Amy and Alice are not synchronized as far as Bob is concerned. Their times differ because of the arbitrarily slow transport of Amy's clock away from Alice. No matter how slow the transport is, Bob will always say that Amy and Alice's clocks are not synchronized.
